COMPLEX NOTATION:

ELECTRIC FIELD $E_x = A\cos(\omega t - kz)$ → (1) → REAL FIELD

COMPLEX NOTATION → $\tilde{E}_x = A \cdot e^{j(\omega t - kz)}$ → (2) → COMPLEX FIELD

LET $\theta = \omega t - kz$

$$\tilde{E}_x = A e^{j\theta} = A \cos\theta + j \sin\theta$$

$$Re\{\tilde{E}_x\} = A\cos\theta = A\cos(\omega t - kz)$$

$$= E_x$$

∴ REAL PART OF THE COMPLEX FIELD = REAL ELECTRIC FIELD

NOTE: YOU CAN ALSO WRITE $\tilde{E}_x = A e^{-j\underbrace{(\omega t - kz)}_{\theta}}$

$$= A e^{-j\theta} = A\cos\theta + j\sin\theta$$

$$Re[\tilde{E}_x] = A\cos\theta = A\cos(\omega t - kz)$$

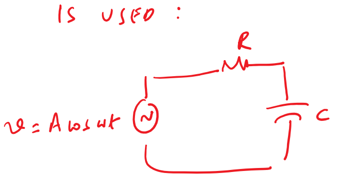
$$= E_x$$

WHY DO WE USE COMPLEX NOTATION?

- LEADS TO MATHEMATICAL SIMPLIFICATIONS.

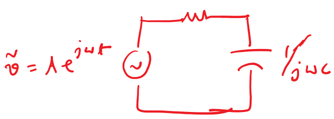
FOR EXAMPLE, WHEN YOU DIFFERENTIATE $e^{j(\omega t - kz)}$ W.R.T. TIME, YOU GET $j\omega e^{j(\omega t - kz)}$, i.e. FUNCTIONAL SHAPE DOES NOT CHANGE. IN CONTRAST, IF YOU DIFFERENTIATE $\cos(\omega t - kz)$, YOU GET $-\omega \sin(\omega t - kz)$.

TYPICALLY, WE EXTEND THE REAL FIELD E_x INTO COMPLEX DOMAIN BY USING EQ. (2). IN CIRCUITS, A SIMILAR PROCEDURE IS USED:



REAL DOMAIN

→



COMPLEX DOMAIN

NOTE THAT ELECTRIC FIELD IS NOT COMPLEX, BUT WE REPRESENT IT IN THE COMPLEX FORM WITH THE UNDERSTANDING THAT THE REAL PART OF THE COMPLEX FIELD CORRESPOND TO THE ACTUAL ELECTRIC FIELD. WE DO ALL THE CALCULATIONS IN COMPLEX DOMAIN & AT THE END, WE TAKE THE REAL PART.

OF $\tilde{E}_x$ TO MAKE CONNECTION TO THE REAL WORLD.

3-D WAVE EQUATION IN SOURCE-FREE REGION

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \longrightarrow (3)$$

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} \longrightarrow (4)$$

$$\nabla \cdot \vec{E} = 0 \longrightarrow (5)$$

HOME WORK : SHOW THAT

$$\nabla^2 \vec{E} - \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \longrightarrow (6)$$

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} \longrightarrow (7)$$

HINT: USE THE VECTOR IDENTITY

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \longrightarrow (8)$$

$$\nabla^2 = \nabla \cdot \nabla$$

TAKE THE CURL OF (3) & USE EQ. (8), EQ. (4) & EQ. (5).

LET $E_x \vec{x} + E_y \vec{y} + E_z \vec{z}$

EQ. (6) IS A VECTOR EQUATION FOR E_x, E_y & E_z. SEPARATELY

EQUATING EACH OF THESE COMPONENTS TO ZERO, WE OBTAIN

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0, \longrightarrow (9) \quad \psi = E_x, E_y \text{ or } E_z.$$

IN THE CASE OF 1-D WAVE EQUATION (REFER TO PREVIOUS LECTURE NOTES), WE HAD

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 E_x}{\partial t^2} = 0 \longrightarrow (10)$$

SO, EQ. (9) IS THE 3-D EXTENSION OF (10).

AS BEFORE, WE TRY A TRIAL SOLUTION

$$\psi = f(t - \alpha_x x - \alpha_y y - \alpha_z z), \quad \alpha_x, \alpha_y \ \& \ \alpha_z \text{ ARE CONSTANTS TO BE DETERMINED.}$$

LET $u = t - \alpha_x x - \alpha_y y - \alpha_z z$

$$\frac{\partial u}{\partial t} = 1, \quad \frac{\partial u}{\partial x} = -\alpha_x, \quad \frac{\partial u}{\partial y} = -\alpha_y, \quad \frac{\partial u}{\partial z} = -\alpha_z$$

PROCEEDING AS BEFORE, YOU FIND

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial u^2}\cdot 1^2, \quad \frac{\partial^2 \psi}{\partial x^2} = \alpha_x^2 \frac{\partial^2 \psi}{\partial u^2}, \quad \frac{\partial^2 \psi}{\partial y^2} = \alpha_y^2 \frac{\partial^2 \psi}{\partial y^2}$$

$$\frac{\partial^2 \psi}{\partial z^2} = \alpha_z^2 \frac{\partial^2 \psi}{\partial u^2}$$

NOW, EQ. (9) BECOMES
$$\left(\alpha_x^2+\alpha_y^2+\alpha_z^2\right)\cancel{\frac{\partial^2\psi}{\partial u^2}} = \frac{1}{v^2}\cancel{\frac{\partial^2\psi}{\partial u^2}}$$

$$\boxed{\alpha_x^2+\alpha_y^2+\alpha_z^2 = \frac{1}{v^2}} \longrightarrow (11)$$

IF WE CHOOSE $f(\;)$ TO BE $A\cos[\omega(\;)]$

$$\psi = f(t-\alpha_x x-\alpha_y y-\alpha_z z)$$
$$= A\cos[\omega(t-\alpha_x x-\alpha_y y-\alpha_z z)]$$
$$= A\cos[\omega t-k_x x-k_y y-k_z z] \longrightarrow (12)$$

3-D PLANE WAVE

$k_x = \omega\alpha_x$, $k_y=\omega\alpha_y$ & so on.

DEFINE A VECTOR, $\vec{k} = k_x\vec{x}+k_y\vec{y}+k_z\vec{z}$

$\vec{k}$ IS KNOWN AS THE <u>WAVE VECTOR</u>.

LET $\vec{r} = x\vec{x}+y\vec{y}+z\vec{z}$

$$\vec{k}\cdot\vec{r} = k_x x+k_y y+k_z z$$

EQ. (12): $\psi = A\cos(\omega t-\vec{k}\cdot\vec{r})$

FROM (11), WE HAVE
$$\alpha_x^2+\alpha_y^2+\alpha_z^2 = \frac{1}{v^2}$$
$$\left(\frac{k_x}{\omega}\right)^2+\left(\frac{k_y}{\omega}\right)^2+\left(\frac{k_z}{\omega}\right)^2 = \frac{1}{v^2}$$
$$k_x^2+k_y^2+k_z^2 = \frac{\omega^2}{v^2}$$

$$\vec{k} = k_x\vec{x}+k_y\vec{y}+k_z\vec{z}, \quad |\vec{k}|^2 = k^2 = k_x^2+k_y^2+k_z^2$$

$$\therefore \quad k^2 = \frac{\omega^2}{v^2} \quad \text{or} \quad \frac{\omega}{k} = \pm v \longrightarrow (13)$$

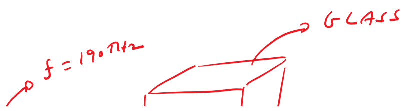
$\frac{\omega}{k} = +v$ $\Rightarrow$ FORWARD PROP. WAVE

$\frac{\omega}{k} = -v$ $\Rightarrow$ BACKWARD PROPAGATING WAVE

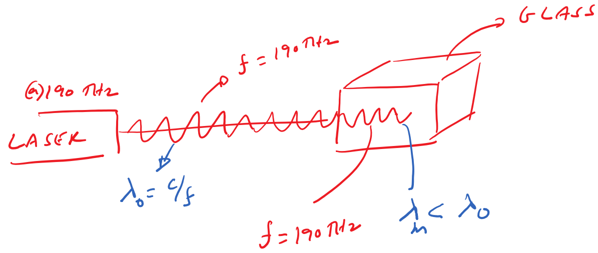
k IS THE <u>WAVE NUMBER</u>.

THE ANGULAR FREQ. $\omega = 2\pi f$ IS DETERMINED BY THE LIGHT SOURCE, SUCH AS LASER OR LED. FOR EXAMPLE, IF THE LASER OPERATES AT 190 THz, THE EM WAVE PROPAGATING IN A DIELECTRIC MEDIUM (ASSUMED TO BE LINEAR) HAS A FREQUENCY, 190 THz. IN A LINEAR MEDIUM, THE FREQUENCY OF THE LAUNCHED EM WAVE CANNOT BE CHANGED.

@190 THz



CHANGED.



CONSIDER A LASER OPERATING AT 190 THz. ITS OUTPUT PROPAGATES IN FREE SPACE & THEN THROUGH GLASS WITH REFRACTIVE INDEX, n.

IN FREE SPACE, $\lambda_0 f = c \longrightarrow (14)$

IN GLASS, LET $\lambda = \lambda_m$. FREQ. DOES NOT CHANGE

$$\therefore \lambda_m f = v \longrightarrow (15)$$

SPEED OF LIGHT IN GLASS

SINCE $v = c/n$, FROM (15) & (14), WE HAVE

$$\frac{\lambda_m f}{\lambda_0 f} = \frac{v}{c} = \frac{\cancel{c}/n}{\cancel{c}} = 1/n$$

$$\boxed{\lambda_m = \frac{\lambda_0}{n}} \longrightarrow (16)$$

SINCE $n > 1$, $\lambda_m < \lambda_0$.

WE HAVE SEEN BEFORE, $k = \frac{2\pi}{\lambda}$

IN FREESPACE, LET $k_0 = \frac{2\pi}{\lambda_0}$

IN GLASS, LET $k_m = \frac{2\pi}{\lambda_m}$

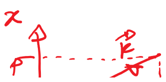
FROM (16), $\frac{k_0}{k_m} = \frac{\cancel{2\pi}/\lambda_0}{\cancel{2\pi}/\lambda_m} = \frac{1}{n}$

or

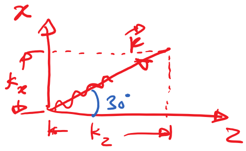
$$\boxed{k_m = k_0 n}$$

THE WAVE NUMBER IN A MEDIUM INCREASES BY A FACTOR n AS COMPARED TO FREESPACE.

EXAMPLE: A LIGHTWAVE WITH ITS ELECTRIC FIELD AMPLITUDE 2V/m AND OF FREQUENCY 250 THz PROPAGATES AS A PLANE WAVE IN GLASS IN THE X-Z PLANE MAKING AN ANGLE OF 30° WITH THE Z-AXIS. WRITE AN EQUATION FOR THIS PLANE WAVE. IGNORE Y-AXIS. ASSUME $n = 1.5$ FOR GLASS.



IGNORE Y-AXIS. ASSUME $n = 1.5$ FOR ...



$$\psi = A\cos(\omega t - k_x x - k_z z)$$

$$A = 2\ V/m$$

$$\omega = 2\pi f = 2\pi \times 250 \times 10^{12}\ \text{rad/s}$$
$$= 1.57 \times 10^{15}\ \text{rad/s}$$

$$v = c/n = \frac{3\times 10^{8}\ m/s}{1.5} = 2\times 10^{8}\ m/s$$

$$\lambda_m f = v \Rightarrow \lambda_m = \frac{2\times 10^{8}}{250\times 10^{12}} = 0.8\ \mu m$$

MICRO

$$k = k_m = \frac{2\pi}{\lambda_m} = 7.854\times 10^{6}\ \text{rad/m}$$

NOTE THAT THE PROPAGATION OF LIGHTWAVE IS IN THE DIRECTION OF $\vec{k}$. THE x- & z- COMPONENTS OF $\vec{k}$ ARE (SEE THE FIGURE)

$$k_z = k\cos 30^\circ = k\sqrt{3}/2$$
$$k_x = k\sin 30^\circ = k/2$$

$$\psi = A\cos(\omega t - k_x x - k_z z)$$
$$= 2\cos\left[1.57\times 10^{15} t - 7.854\times 10^{6}\left(\frac{\sqrt{3}}{2}z + x/2\right)\right]\ V/m$$

HELMHOLTZ EQUATION:

THE ELECTRIC FIELD IS TIME-VARYING AS WELL AS SPACE-VARYING. MOST OF THE OPTICAL SOURCES (SUCH AS LASER) HAVE SINUSOIDAL TIME DEPENDENCE. TO SEPARATE THE SPATIAL PART FROM THE TEMPORAL PART, WE WRITE

$$\psi(x,y,z,t) = \varphi(x,y,z)\cdot e^{j\omega t}$$

WAVE EQUATION:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

$$\frac{\partial\psi}{\partial t} = j\omega\cdot\varphi(x,y,z)\cdot e^{j\omega t}$$

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2\varphi(x,y,z)\, e^{j\omega t}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\varphi}{\partial x^2} e^{j\omega t} \quad \& \text{ so on}$$

∴ WAVE EQUATION BECOMES

$$\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}\right)e^{j\omega t}+\frac{\omega^2}{v^2}\phi e^{j\omega t}=0$$

$$\boxed{\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}+k^2\varphi=0}$$

or

$$\boxed{\nabla^2\varphi+k^2\varphi=0}$$

HELMHOLTZ EQUATION

HELMHOLTZ EQUATION FORMS THE BASIS FOR ANALYZING THE FIBER MODES. WE WILL DISCUSS THIS FURTHER WHEN WE STUDY OPTICAL FIBERS