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Q1) Find the optimum of the following function:

$$f(x,y) = 2xy + 1.5y - 1.25x^2 - 2y^2 + 5$$

Solution:

$$\frac{\partial f}{\partial x} = 2y - 2.5x = 0$$

$$\frac{\partial f}{\partial y} = 2x + 1.5 - 4y = 0$$

$$\Rightarrow \begin{cases} 2y - 2.5x = 0 \\ -4y + 2x = -1.5 \end{cases} \Rightarrow x = 0.5 \ \& \ y = 0.4$$

or $(0.5, 0.4) = (x^*, y^*)$

$$\frac{\partial^2 f}{\partial x^2} = -2.5, \quad \frac{\partial^2 f}{\partial y^2} = -4$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 2$$

$$\Rightarrow H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

$$\Rightarrow H = \begin{bmatrix} -2.5 & 2 \\ 2 & -4 \end{bmatrix} \Rightarrow |H| = 4 * 2.5 - 4 = 6 > 0$$

Since $|H| > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, Point $(0.5, 0.4)$ is a local Maximum of $f(x,y)$

Q2) Find the minimum value of

$$f(x,y) = (x-3)^2 + (y-2)^2$$

Starting at $x=1$ and $y=1$, using the steepest descent method with a stopping criterion of $\epsilon_s = 1\%$.

Solution: $\frac{\partial f}{\partial x} = 2(x-3)$, $\frac{\partial f}{\partial y} = 2(y-2)$

$$\Rightarrow \vec{\nabla f} = 2(x-3)\vec{i} + 2(y-2)\vec{j}$$

$$(x_0, y_0): \quad g(\alpha) = f\left(x_0 + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot \alpha ,\ y_0 + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot \alpha\right)$$

$$= f\left(x_0 + 2(x_0-3)\cdot\alpha ,\ y_0 + 2(y_0-2)\alpha\right)$$

$$= (x_0 + 2(x_0-3)\alpha - 3)^2 + (y_0 + 2(y_0-2)\alpha - 2)^2 , \quad x_0 = 1,\ y_0 = 1$$

$$= (-2-4\alpha)^2 + (-1-2\alpha)^2$$

$$\Rightarrow g'(\alpha) = 2(-4)(-2-4\alpha) + 2(-2)(-1-2\alpha) = 0$$

$$\Rightarrow -8(-2-4\alpha) = 4(-1-2\alpha) \Rightarrow \boxed{\alpha = -0.5}$$

$$\alpha^* = \alpha = -0.5$$

$$\Rightarrow x_1 = x_0 + 2(x_0-3)\alpha^* = 1 - 4(-0.5) = 1+2 = 3$$

$$y_1 = y_0 + 2(y_0-2)\alpha^* = 1 + 2(1-2)(-0.5) = 1+1 = 2$$

$\Rightarrow x_1 = 3$, $y_1 = 2$, and we iterate using these new values for x and y ....

Q3) Use the Newton's Search method to find the minimum of $f(x,y) = x^2 - 4x + y^2 - y - xy$.

Start at $(1,2)$.

Solution:

In Newton's method: $\vec{X}_{i+1} = \vec{X}_i - H_i^{-1} \nabla f(\vec{X}_i)$

$$\vec{\nabla} f = (2x - 4 - y)\vec{i} + (2y - 1 - x)\vec{j} \quad \text{or} \quad \vec{\nabla} f = \begin{pmatrix} 2x-4-y \\ 2y-1-x \end{pmatrix}$$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow H^{-1} = \frac{1}{4-1}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow H^{-1}.\nabla f = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2x-4-y \\ 2y-1-x \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4x-8-2y+2y-1-x \\ 2x-4-y+4y-2-2x \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3x-9 \\ 3y-6 \end{bmatrix} = \begin{bmatrix} x-3 \\ y-2 \end{bmatrix}$$

$$\Rightarrow \vec{X}_{i+1} = \vec{X}_i - H_i^{-1}\nabla f(x_i) = \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} x_i - 3 \\ y_i - 2 \end{bmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

We could converge without any iteration. !