Tutorial 2 of 3SK3

Maryam Mohseni

Question 1: Bisection

Find a solution to $f(x) = x^2 - 3 = 0$, with percent relative error $\leq 1\%$. Bisection method: $x_r = \dfrac{x_l + x_u}{2}$.

Solution

1) Select x_l and x_u such that $f(x_l) \cdot f(x_u) < 0$. Thus, values for x_l and x_u can be found:

$$\left.\begin{array}{l} x_l = 1 \Rightarrow f(x_l) = 1^2 - 3 = -2 \\ x_u = 2 \Rightarrow f(x_u) = 2^2 - 3 = 1 \end{array}\right\} \Rightarrow f(1) \cdot f(2) < 0.$$

2) Now, using $x_l = 1$ and $x_u = 2$, estimate the solution, x_r, as $x_r = \dfrac{1+2}{2} = 1.5.$

Therefore, $f(x_r) = \left(\dfrac{3}{2}\right)^2 - 3 = -\dfrac{3}{4}$. $\because f(x_r) \cdot f(x_u) < 0$, let $x_l = x_r = 1.5$.

Question 1: Bisection

3) Now, $x_l = 1.5$, $x_u = 2$. Repeat step 2 for new values of x_l and x_u. Now, $x_u = x_r = 1.75$. Check relative error:

$$\varepsilon_r = \left| \frac{x_r^{\text{new}} - x_r^{\text{old}}}{x_r^{\text{new}}} \right| = \frac{1.75 - 1.5}{1.75} = 0.14 = 14\% > 1\%.$$

4) Repeat step 3 until $x_l = 1.7188$, $x_u = 1.75$. Now, $x_r = \frac{1}{2}(1.7188 + 1.75) = 1.7344$.

Check relative error:

$$\varepsilon_r = \left| \frac{x_r^{\text{new}} - x_r^{\text{old}}}{x_r^{\text{new}}} \right| = \frac{1.7344 - 1.7188}{1.7344} = 0.0089 = 0.89\% < 1\%.$$

Thus, the stopping criterion is satisfied. $\therefore x_r = 1.7344$.

Question 1: Newton-Raphson

Find a solution to $f(x) = x^2 - 3 = 0$, with percent relative error $\leq 1\%$. Newton-Raphson method: $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$.

Solution

1) $\because f(x) = x^2 - 3,\ \ f'(x) = 2x.$

2) Let $x_0 = 1$. $\therefore f(x_0) = -2,\ \ f'(x_0) = 2$ and $x_1 = 1 - \dfrac{-2}{2} = 2.$

3) Now, $x_1 = 2$. Repeat step 2 for $x_1 = 2$. $\therefore x_2 = 1.75$. Check relative error:

$$\varepsilon_r = \left|\frac{x_2 - x_1}{x_2}\right| = \frac{|1.75 - 2|}{1.75} = 0.1429 = 14.29\% > 1\%.$$

Question 1: Newton-Raphson

4) Repeat step 3 until $x_5 = 1.7321$. $\therefore f(x_4) = 3.1088 \times 10^{-4}$, $f^{'}(x_4) = 3.4643$ and

$$x_5 = 1.7321 - \frac{3.1088 \times 10^{-4}}{3.4643} = 1.7321.$$ Check relative error:

$$\varepsilon_r = \left| \frac{x_5 - x_4}{x_5} \right| = \frac{|1.7321 - 1.7321|}{1.7321} = 0 < 1\%.$$

Thus, the stopping criterion is satisfied. The root is $x_5 = 1.7321$.

Sample Code:

```
%Matlab Code for Newton-Raphson Method solution to f(x)=x^2-3=0
%percentage error must be less than 1. Note: final answer is: x=1.7321
x1=1; % initial point
error=1; % initial value of error is 100%.
n=1;
while error>0.01
x2=x1-(x1^2-3)/(2*x1)
n=n+1
error=abs((x1-x2)/x2)
x1=x2;
end
x=x2 % solution
n % iternation number
```

Question 1: Secant

Find a solution to $f(x) = x^2 - 3 = 0$, with percent relative error $\leq 1\%$. Secant method: $x_{i+1} = x_i - \dfrac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$.

Solution

1) Let $x_0 = 1$ and $x_1 = 2$. $\because f(x) = x^2 - 3$, $\therefore f(x_0) = -2$ and $f(x_1) = 1$.

2) Now, using $x_0 = 1$ and $x_1 = 2$, $x_2 = 2 - \dfrac{2-1}{1-(-2)} = 1.667$. $\therefore f(x_2) = -0.2222$.

Question 1: Secant

3) Now, $x_1 = 2$ and $x_2 = 1.6667$. Repeat step 2 for $x_1 = 2$ and $x_2 = 1.6667$.

$\therefore x_3 = 1.6667 - \dfrac{-0.2222(1.6667-2)}{-0.2222-1} = 1.7273$. $\therefore f(x_3) = -0.0165$. Check relative error:

$$\varepsilon_r = \left|\frac{x_3 - x_2}{x_3}\right| = \frac{|1.7273 - 1.6667|}{1.7273} = 0.0351 = 3.51\% > 1\%.$$

4) Repeat step 3 until $x_4 = 1.7321$ and $x_3 = 1.7273$. $f(x_4) = 0.0003$.

$\therefore x_5 = 1.7321 - \dfrac{0.0003(1.7321-1.7273)}{0.0003-(-0.0165)} = 1.7321$. Check relative error:

$$\varepsilon_r = \left|\frac{x_5 - x_4}{x_5}\right| = \frac{|1.7321 - 1.7321|}{1.7321} = 0 < 1\%.$$

Thus, the stopping criterion is satisfied. The root is $x_5 = 1.7321$.

Question 2

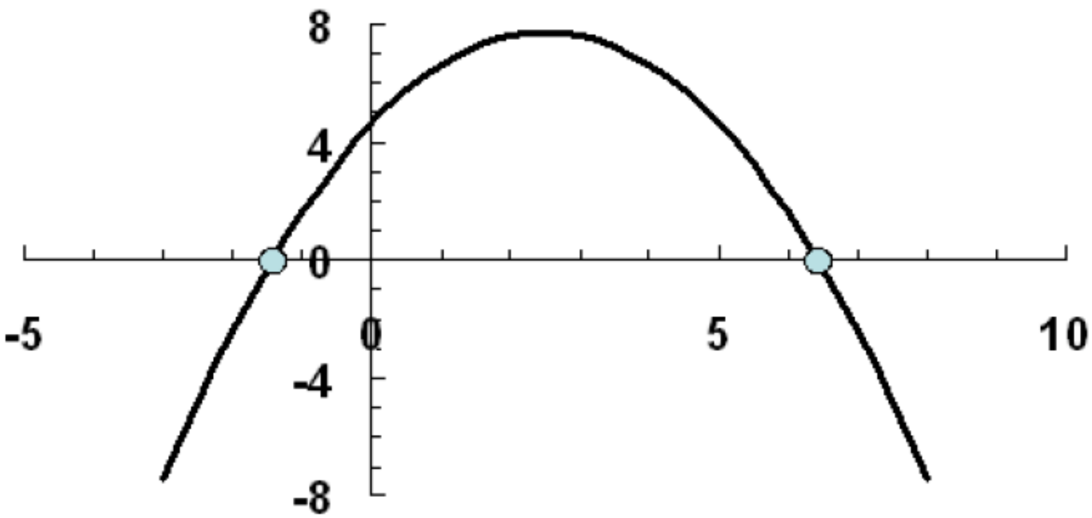
- Determine the real roots of

  $f(x)=-0.5x^2+2.5x+4.5$

1) Graphically
2) Using the quadratic formula
3) Using three iterations of the bisection method to determine the highest root. Employ initial guesses of $x_l=5$ and $x_u=10$ Compute the estimated error and the true error after each iteration.

Question 2

1) A plot indicates that roots occur at about $x = -1.4$ and 6.4.



Question 2

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2.5 \pm \sqrt{(2.5)^2 - 4(-0.5)(4.5)}}{2(-0.5)} = \begin{matrix} -1.40512 \\ 6.40512 \end{matrix}$$

Question 2

- First iteration:

$$x_r = \frac{5+10}{2} = 7.5$$

$$\varepsilon_t = \left|\frac{6.40512 - 7.5}{6.40512}\right| \times 100\% = 17.09\% \qquad \varepsilon_a = \left|\frac{10-5}{10+5}\right| \times 100\% = 33.33\%$$

$$f(5)f(7.5) = 4.5(-4.875) = -21.9375$$

Therefore, the bracket is $x_l = 5$ and $x_u = 7.5$.

Question 2

Second iteration:

$$x_r = \frac{5 + 7.5}{2} = 6.25$$

$$\varepsilon_t = \left|\frac{6.40512 - 6.25}{6.40512}\right| \times 100\% = 2.42\% \qquad \varepsilon_a = \left|\frac{7.5 - 5}{7.5 + 5}\right| \times 100\% = 20.00\%$$

$$f(5)f(6.25) = 4.5(0.59375) = 2.672$$

Consequently, the new bracket is $x_l = 6.25$ and $x_u = 7.5$.

Question 2

- Third iteration

$$x_r = \frac{6.25 + 7.5}{2} = 6.875$$

$$\varepsilon_t = \left|\frac{6.40512 - 6.875}{6.40512}\right| \times 100\% = 7.34\%$$

$$\varepsilon_a = \left|\frac{7.5 - 6.25}{7.5 + 6.25}\right| \times 100\% = 9.09\%$$

Thank you