Tutorial 3 of 3SK3

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Question 1: Gaussian elimination rounding error

Solve the problem

$$\left[\begin{array}{cc|c} -0.0590 & 0.2372 & -0.3528 \\ 0.1080 & -0.4348 & 0.6452 \end{array}\right]\begin{array}{c}(1)\\(2)\end{array},$$

supposing that the algorithm uses only 4 significant digits for the calculation. (Note: exact solution is $x_1 = 10, x_2 = 1$.

Solution

1) The multiplier for the second row is:

$$\frac{0.1080}{-0.0590} = -1.830518\cdots \approx 1.831, \text{ to 4 significant digits.}$$

2) The second entry of the second matrix row becomes:

$$-0.4308 - (-1.831)(0.2372) = -0.4348 + 0.4343 = -0.0005.$$

3) The third entry of the second row becomes:

$$0.6452 - (-1.831)(-0.3528) = -0.64520 - 0.6460 = -0.0008.$$

Question 1: Gaussian elimination rounding error

Thus, the system of equations becomes

$$\left[\begin{array}{cc|c} -0.0590 & 0.2372 & -0.3528 \\ 0 & -0.0005 & -0.0008 \end{array}\right]\begin{array}{c}(3)\\(4)\end{array}.$$

Therefore, from Eqs.(3) and (4):

$$\begin{cases} \widehat{x_2} = 1.600 \\ \widehat{x_1} = \left(-0.3528 - 0.2372 \cdot 1.600\right) / -0.0590 = 12.41 \end{cases}.$$

As one can see, there is significant error compared to the true solution.

Question 2: Gaussian elimination pivoting

Solve the linear equation

$$\begin{cases} \varepsilon x_1 + x_2 = 1 \ (1) \\ x_1 + x_2 = 2 \ \ (2) \end{cases}.$$

Solution

1) From Eqs.(1) and (2),

$$(1)\times\left(-\frac{1}{\varepsilon}\right)+(2):\left(1-\frac{1}{\varepsilon}\right)x_2 = 2-\frac{1}{\varepsilon}\ (3).$$

Therefore, from Eqs.(1) and (3),

$$\begin{cases} x_2 = \left(2-\frac{1}{\varepsilon}\right)\Big/\left(1-\frac{1}{\varepsilon}\right) \\ x_1 = (1-x_2)/\varepsilon \end{cases}.$$

Question 2: Gaussian elimination pivoting

If ε is very small, then $1/\varepsilon$ is very large compared to 1, and with rounding

$$\begin{cases} x_2 \doteq -\frac{1}{\varepsilon} \Big/ -\frac{1}{\varepsilon} = 1 \\ x_1 = (1-1)/\varepsilon = 0 \end{cases},$$

which does not satisfy Eq.(2).

2) Now, if the order of the equations is changed, solve

$$\begin{cases} x_1 + x_2 = 2 \quad (2) \\ \varepsilon x_1 + x_2 = 1 \ (1) \end{cases}.$$

Question 2: Gaussian elimination pivoting

From Eqs.(1) and (2),

$$(2)\times(-\varepsilon)+(1):(1-\varepsilon)x_2=1-2\varepsilon\ (4).$$

Thus, from Eqs.(2) and (4),

$$\begin{cases} x_2=(1-2\varepsilon)/(1-\varepsilon) \\ x_1=2-x_2 \end{cases}.$$

Therefore, after changing the order of the equations, there are no rounding problems.

Question 3: LU decomposition

- Without pivoting strategy find L and U matrices

$$
\begin{aligned}
2x_1 + x_2 - x_3 + 2x_4 &= 5 \\
4x_1 + 5x_2 - 3x_3 + 6x_4 &= 9 \\
-2x_1 + 5x_2 - 2x_3 + 6x_4 &= 4 \\
4x_1 + 11x_2 - 4x_3 + 8x_4 &= 2
\end{aligned}
$$

$$
[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 4 & 5 & -3 & 6 & 9 \\ -2 & 5 & -2 & 6 & 4 \\ 4 & 11 & -4 & 8 & 2 \end{array}\right]
$$

Question 3: LU decomposition

When eliminate a_{21}, a_{31}, a_{41}, we need to use $m_{2,1} = 2$, $m_{3,1} = -1$, $m_{4,1} = 2$ times a_{11}, then we will get

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 6 & -3 & 8 & 9 \\ 0 & 9 & -2 & 4 & -8 \end{array}\right]$$

Question 3: LU decomposition

- When eliminate a_{32}, a_{42}, we need to use $m_{3,2} = 2$, $m_{4,2} = 3$ times a_{22}, then we will get

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right]$$

Question 3: LU decomposition

When eliminate a_{43}, we need to use $m_{4,3} = -1$ times a_{33}, then we will get

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right]$$

$$\begin{aligned} 2x_1 + x_2 - x_3 + 2x_4 &= 5 \\ 3x_2 - x_3 + 2x_4 &= -1 \\ -x_3 + 4x_4 &= 11 \\ 2x_4 &= 6 \end{aligned}$$

Question 3: LU decomposition

$$U = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & 0 \\ m_{4,1} & m_{4,2} & m_{4,3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 3 & -1 & 1 \end{bmatrix}$$

Thank you