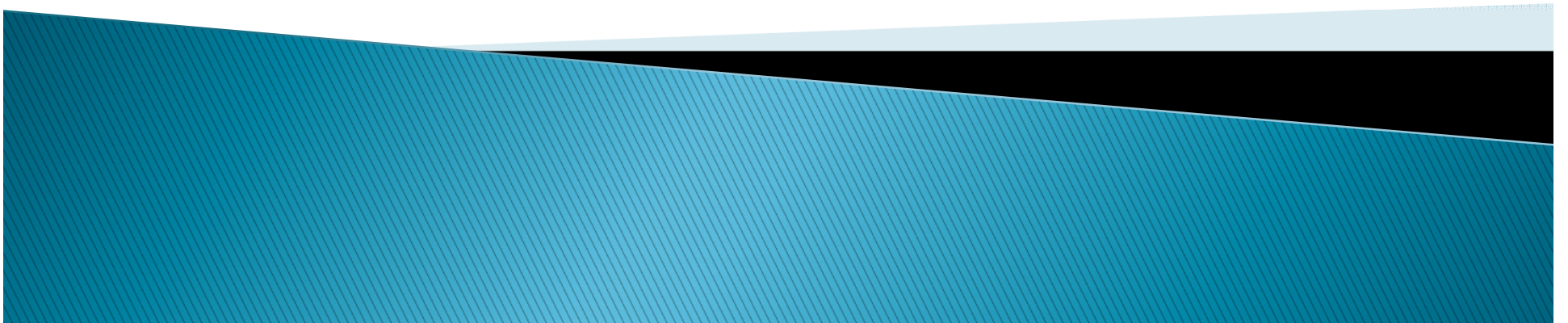


Tutorial 3 of 3SK3

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Question 1: Gaussian elimination rounding error

Solve the problem

$$\left[\begin{array}{cc|c} -0.0590 & 0.2372 & -0.3528 \\ 0.1080 & -0.4348 & 0.6452 \end{array} \right] \begin{matrix} (1) \\ (2) \end{matrix}$$

supposing that the algorithm uses only 4 significant digits for the calculation. (Note: exact solution is $x_1 = 10, x_2 = 1$.)

Solution

- 1) The multiplier for the second row is:

$$\frac{0.1080}{-0.0590} = -1.830518\dots \approx \bar{1.831}, \text{ to 4 significant digits.}$$

- 2) The second entry of the second matrix row becomes:

$$-0.4308 - (-1.831)(0.2372) = -0.4348 + 0.4343 = -0.0005.$$

- 3) The third entry of the second row becomes:

$$0.6452 - (-1.831)(-0.3528) = -0.64520 - 0.6460 = -0.0008.$$



Question 1: Gaussian elimination rounding error

Thus, the system of equations becomes

$$\begin{cases} \left[\begin{array}{cc|c} -0.0590 & 0.2372 & -0.3528 \end{array} \right] (3) \\ \left[\begin{array}{cc|c} 0 & -0.0005 & -0.0008 \end{array} \right] (4) \end{cases}$$

Therefore, from Eqs.(3) and (4):

$$\begin{cases} \hat{x}_2 = 1.600 \\ \hat{x}_1 = (-0.3528 - 0.2372 \cdot 1.600) / -0.0590 = 12.41 \end{cases}$$

As one can see, there is significant error compared to the true solution.



Question 2: Gaussian elimination pivoting

Solve the linear equation

$$\begin{cases} \varepsilon x_1 + x_2 = 1 & (1) \\ x_1 + x_2 = 2 & (2) \end{cases}$$

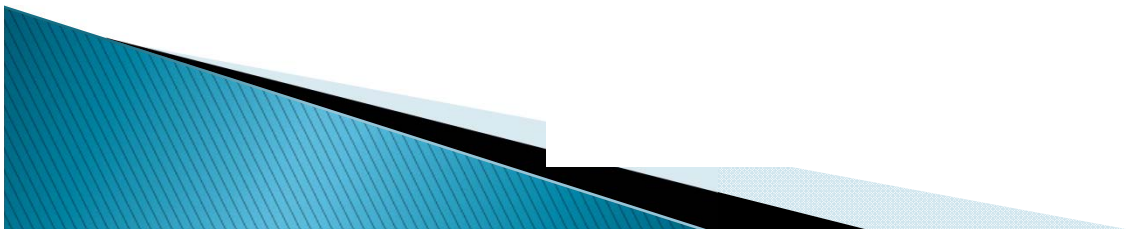
Solution

1) From Eqs.(1) and (2),

$$(1) \times \left(-\frac{1}{\varepsilon}\right) + (2) : \left(1 - \frac{1}{\varepsilon}\right) x_2 = 2 - \frac{1}{\varepsilon} \quad (3).$$

Therefore, from Eqs.(1) and (3),

$$\begin{cases} x_2 = \left(2 - \frac{1}{\varepsilon}\right) / \left(1 - \frac{1}{\varepsilon}\right) \\ x_1 = (1 - x_2) / \varepsilon \end{cases}$$



Question 2: Gaussian elimination pivoting

If ε is very small, then $1/\varepsilon$ is very large compared to 1, and with rounding

$$\begin{cases} x_2 \doteq -\frac{1}{\varepsilon} / -\frac{1}{\varepsilon} = 1 \\ x_1 = (1-1)/\varepsilon = 0 \end{cases},$$

which does not satisfy Eq.(2).

2) Now, if the order of the equations is changed, solve

$$\begin{cases} x_1 + x_2 = 2 & (2) \\ \varepsilon x_1 + x_2 = 1 & (1) \end{cases}.$$



Question 2: Gaussian elimination pivoting

From Eqs.(1) and (2),

$$(2) \times (-\varepsilon) + (1) : (1 - \varepsilon)x_2 = 1 - 2\varepsilon \quad (4).$$

Thus, from Eqs.(2) and (4),

$$\begin{cases} x_2 = (1 - 2\varepsilon)/(1 - \varepsilon) \\ x_1 = 2 - x_2 \end{cases}.$$

Therefore, after changing the order of the equations, there are no rounding problems.



Question 3: LU decomposition

- ▶ Without pivoting strategy find L and U matrices

$$2x_1 + x_2 - x_3 + 2x_4 = 5$$

$$4x_1 + 5x_2 - 3x_3 + 6x_4 = 9$$

$$-2x_1 + 5x_2 - 2x_3 + 6x_4 = 4$$

$$4x_1 + 11x_2 - 4x_3 + 8x_4 = 2$$

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 4 & 5 & -3 & 6 & 9 \\ -2 & 5 & -2 & 6 & 4 \\ 4 & 11 & -4 & 8 & 2 \end{array} \right]$$

Question 3: LU decomposition

When eliminate a_{21} , a_{31} , a_{41} , we need to use $m_{2,1} = 2$, $m_{3,1} = -1$, $m_{4,1} = 2$ times a_{11} , then we will get

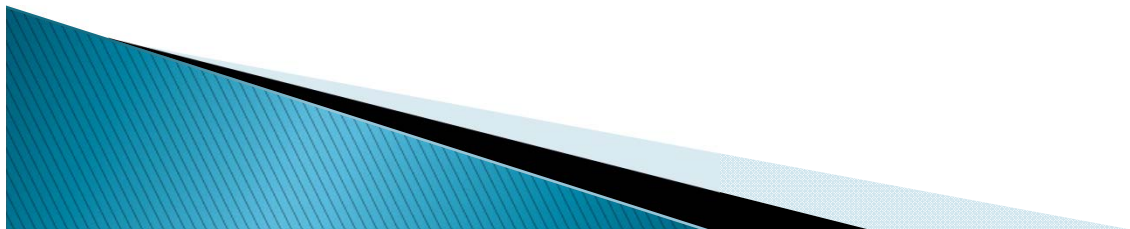
$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 6 & -3 & 8 & 9 \\ 0 & 9 & -2 & 4 & -8 \end{array} \right]$$



Question 3: LU decomposition

- ▶ When eliminate a_{32} , a_{42} , we need to use $m_{3,2} = 2$, $m_{4,2} = 3$ times a_{22} , then we will get

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 1 & -2 & -5 \end{array} \right]$$



Question 3: LU decomposition

When eliminate a_{43} , we need to use $m_{4,3} = -1$ times a_{33} , then we will get

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 0 & 2 & 6 \end{array} \right]$$

$$2x_1 + x_2 - x_3 + 2x_4 = 5$$

$$3x_2 - x_3 + 2x_4 = -1$$

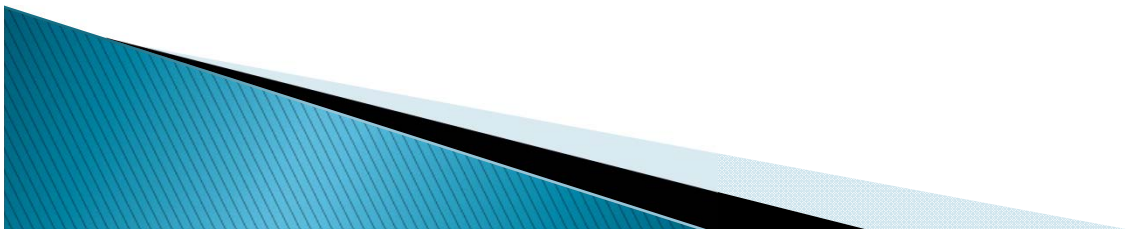
$$-x_3 + 4x_4 = 11$$

$$2x_4 = 6$$



Question 3: LU decomposition

$$U = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & 0 \\ m_{4,1} & m_{4,2} & m_{4,3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 3 & -1 & 1 \end{bmatrix}$$



Thank you

