Constrained Optimization
Chapter 15

LINEAR PROGRAMMING

- An optimization approach that deals with meeting a desired objective such as maximizing profit or minimizing cost in presence of constraints such as limited resources
- Mathematical functions representing both the objective and the constraints are linear.

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Standard Form/

- Basic linear programming problem consists of two major parts:
  - The objective function
  - A set of constraints
- For maximization problem, the objective function is generally expressed as

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

c_j= payoff of each unit of the *j*th activity that is undertaken

x_j= magnitude of the *j*th activity

Z= total payoff due to the total number of activities

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- The constraints can be represented generally as

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$$

- Where a_{ij}=amount of the ith resource that is consumed for each unit of the *j*th activity and b_i=amount of the *i*th resource that is available
- The general second type of constraint specifies that all activities must have a positive value, x_i>0 .
- Together, the objective function and the constraints specify the linear programming problem.

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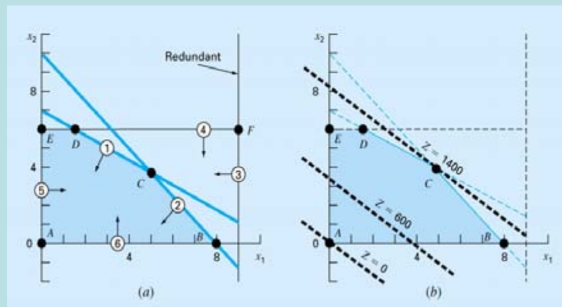
Figure 15.1



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Possible outcomes that can be generally obtained in a linear programming problem/

1. *Unique solution*. The maximum objective function intersects a single point.
2. *Alternate solutions*. Problem has an infinite number of optima corresponding to a line segment.
3. *No feasible solution*.
4. *Unbounded problems*. Problem is under-constrained and therefore open-ended.

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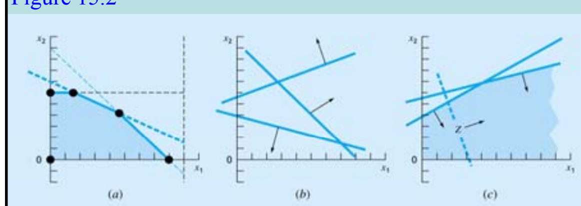
Figure 15.2



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The Simplex Method/

- Assumes that the optimal solution will be an extreme point.
- The approach must discern whether during problem solution an extreme point occurs.
- To do this, the constraint equations are reformulated as equalities by introducing slack variables.

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- A slack variable measures how much of a constrained resource is available, e.g.,

  $7x_1 + 11\,x_2 \le 77$

  If we define a slack variable S_1 as the amount of raw gas that is not used for a particular production level (x_1, x_2) and add it to the left side of the constraint, it makes the relationship exact.

  $7x_1 + 11\,x_2 + S_1 = 77$
- If slack variable is positive, it means that we have some slack that is we have some surplus that is not being used.
- If it is negative, it tells us that we have exceeded the constraint.
- If it is zero, we have exactly met the constraint. We have used up all the allowable resource.

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Maximize

$$Z = 150x_1 + 175x_2$$

$$\begin{aligned}
7x_1 + 11x_2 + S_1 &= 77 \\
10x_1 + 8x_2 + S_2 &= 80 \\
x_1 + S_3 &= 9 \\
x_2 + S_4 &= 6
\end{aligned}$$

$$x_1, x_2, S_1, S_2, S_3, S_4 \ge 0$$

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- We now have a system of linear algebraic equations.
- For even moderately sized problems, the approach can involve solving a great number of equations. For *m* equations and *n* unknowns, the number of simultaneous equations to be solved are:

$$C_m^n = \frac{n!}{m!(n-m)!}$$

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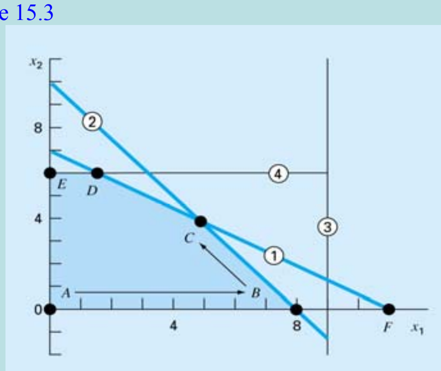
Figure 15.3



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