The Union-Find Problem

The Problem: Given a set X of n elements $x_1, x_2, \ldots, x_n$. We would like to maintain a collection of disjoint subsets (groups) of X.

Initially, the collection is empty.
There are three operations on the elements and the subsets.

Make_set(i): makes x_i a subset and assigns a name for the subset.
Find(i): returns the name of the subset that contains x_i.
Union(i, j): combines subsets that contain x_i and x_j, say S_i and S_j, into a new subset with a unique name. (Any name distinct from other names will do.)

The goal: Design a data structure that will support any sequence of these three operations as efficient as possible.

Note: We assume the types for elements are subrange type. Therefore we can use elements name to index into array (e.g. integer $1, \ldots, n$)

A simple (naive) solution

Store the name of the subset containing the i'th element x_i in $A[i]$.

- **Make_set(i)**: we just set $A[i]$ to i.
- **Find(i)**: we just look at $A[i]$ and find out the name for the subset.
- **Union(i, j)**: (Assume the name of the resulting subset is S_i's name) Change the subset name for all elements in S_j.

Example:

Make_set(1 ... 7)	1	2	3	4	5	6	7
Union(1, 2)	1	1	3	4	5	6	7
Union(5, 6)	1	1	3	4	5	5	7
Find(6)	5						
Union(1, 5)	1	1	3	4	1	1	7
Union(3, 1)	3	3	3	4	3	3	7

Time: n union operations may need $O(n^2)$ time.

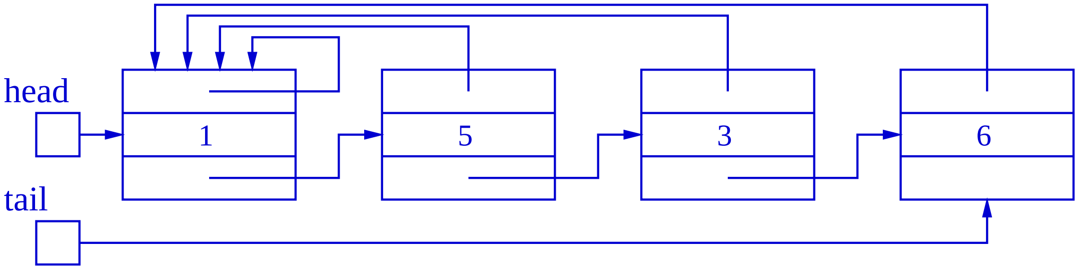
An improved implementation

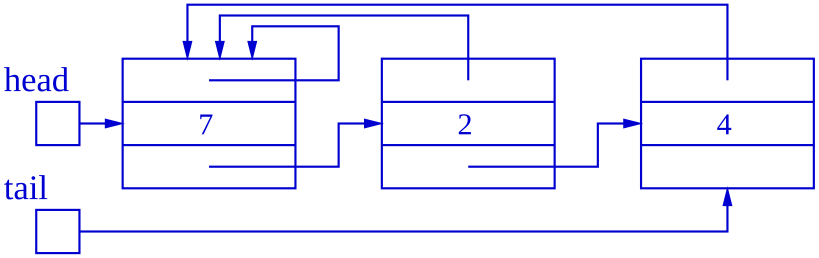
- Each set is represented by a linked list.
- The first node in each list serves as its set's representative.
- Each node of the list contains a set member, a pointer to the next node, and a pointer back to the representative.
- Each list maintains a pointer, head, to the first node and a pointer, tail, to the last node.
- Make_set(i) and Find(i) are easy to implement.
- For the Union(i,j), we will append the smaller list onto the longer list and update representative pointers of the smaller list.

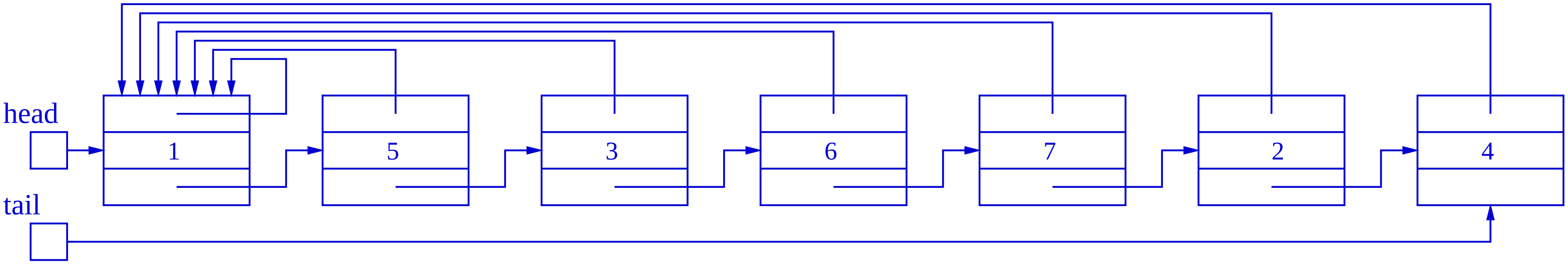
Time: with a sequence of m operations, n of which are Make_set operations, it takes $O(m + n \log n)$ time.

Why: how many times a pointer to its representative can be changed?

Example:







Another implementation

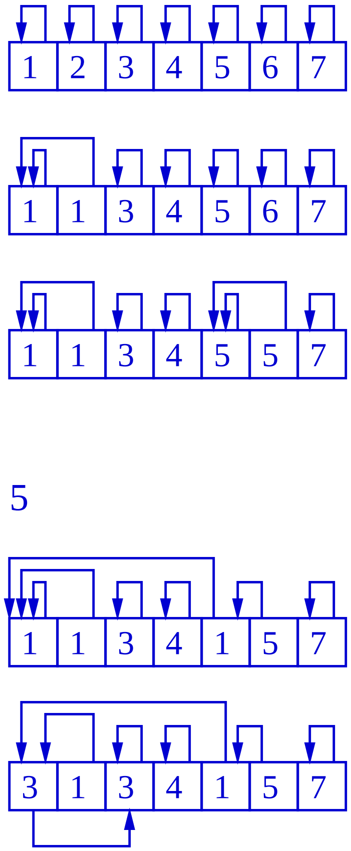
- Instead of making *Find* operation simple, we make *Union* operation simple.
- Each set is a tree and each node in a tree is a record: one field for element name, one field for a pointer (parent pointer) to another node.

† **Find(i)**: from entry i, follow parent pointer until we find a node with a nil pointer (root). Return the name in that node.

† **Union(i, j)**: we change the pointer of the root of set S_j to pointing to the root of set S_i, or vice-versa.

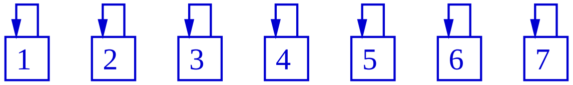
Example:

Make_set(1 ... 7)	1	2	3	4	5	6	7
Union(1, 2)	1	1	3	4	5	6	7
Union(5, 6)	1	1	3	4	5	5	7
Find(6)	5						
Union(1, 5)	1	1	3	4	1	5	7
Union(3, 1)	3	1	3	4	1	5	7

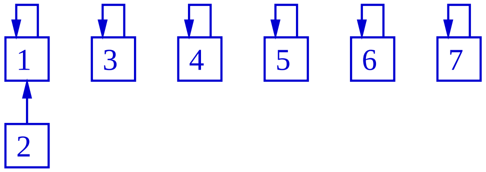


We can consider the whole structure as a forest.

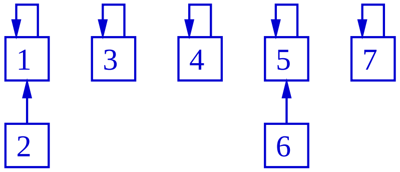
Make_set(1 ... 7)



Union(1, 2)

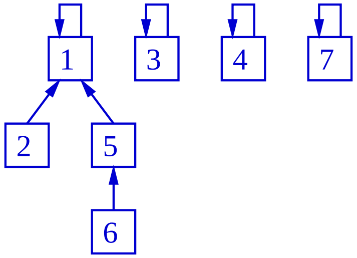


Union(5, 6)

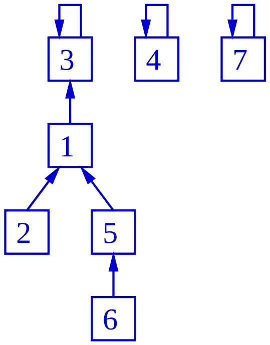


Find(6) 5

Union(1, 5)



Union(3, 1)



Efficient *Union-Find*

- *Idea*: balance and collapse the trees.
- Balancing: when union operation is performed, the root pointer of the smaller tree is set to point to the root of larger tree.
  - † Rather than explicitly keeping the size of the subtree rooted at each node, we use another approach.
  - † For each node, we maintain a **rank** that is an upper bound on the height of that node.
  - † In union by rank, the root with smaller rank is made to point to the root with larger rank during an Union operation.
    - – If two roots have equal ranks, we arbitrarily choose one of the roots as the the parent, increase its rank by 1, and reset the other root.
    - – With Make_set(), the rank is set to 0.

- (1) If union by rank is used, then for any node, its height is bounded by its rank.
- (2) If union by rank is used, then for any node i, its rank is bounded by $\log(size(i))$.

Proof: (of (1)) Induction on the number of Make_set and Union operations.

Base case: the first operation must be Make_set, and (1) is true since we have one node with height 0 and rank 0.
Induction step: consider an $Union(i, j)$ operation and let r_i and r_j be the roots of the trees containing i and j. We assume that $height(r_i) \leq rank[r_i]$ and $height(r_j) \leq rank[r_j]$.

† If $rank[r_i] > rank[r_j]$, $\quad height(union(i, j))$
$= \max\{height(r_i), height(r_j) + 1\}$
$\leq rank[r_i] = rank[root(union(i, j))]$.

† If $rank[r_i] < rank[r_j]$, $\quad height(union(i, j))$
$= \max\{height(r_i) + 1, height(r_j)\}$
$\leq rank[r_j] = rank[root(union(i, j))]$.

† If $rank[r_i] = rank[r_j]$, $\quad height(union(i, j))$
$\leq \max\{height(r_i) + 1, height(r_j) + 1\}$
$\leq rank[r_i] + 1 = rank[root(union(i, j))]$. □

Proof: (of (2)) Induction on the number of Make_set and Union operations.

- With balancing (union by rank), for a sequence of m operations, n of which are Make_set operations, the height of any tree is less than or equal to $\log n$, since we only have n elements.
- Any find operation is at most $O(\log n)$
- Any sequence of $m \geq n$ operations will be bounded by $O(m \log n)$.

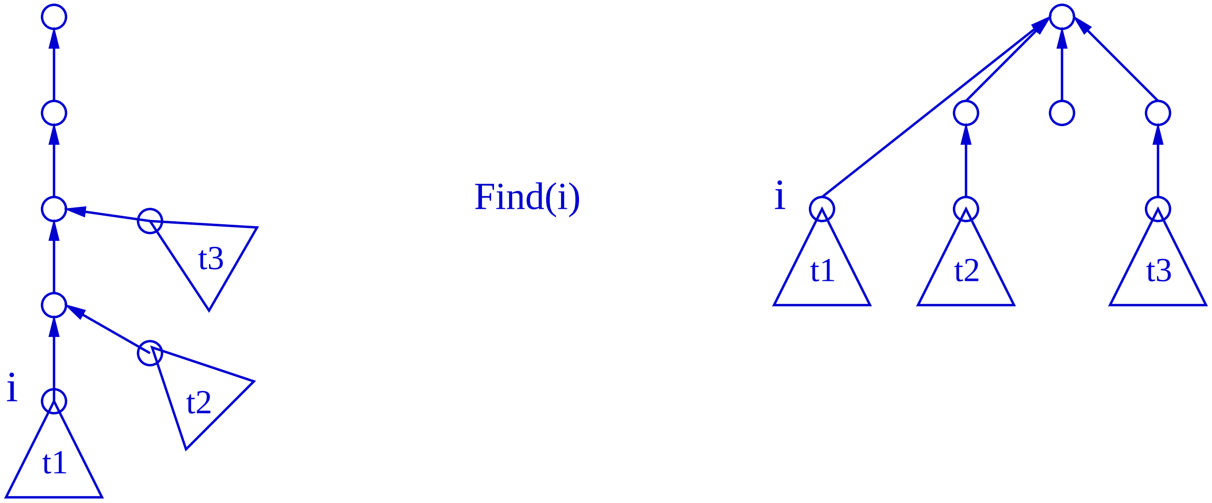
Union: constant time.

Find: $O(\log n)$ time.

Path compression (collapse the tree)

In the operation of *Find(i)*, do following:

first pass: follow parent pointer to find the root
second pass: follow parent pointer and change each of the pointers in the path to point to root.



With path compression alone, for a sequence of m operations, n of which are Make_set operations, the time complexity is $O(m \log n)$.

Theorem. If both balancing and path comparisons are used, then the total number of steps in the worst case for any sequence of $m \geq n$ operations, n of which are Make_set operations, is $O(m \log^* n)$.

Proof: Omitted.

$\log^*(1) = 0$, $\log^*(2) = 1$.
$\log^*(n) = 1 + \log^*(\lceil \log_2 n \rceil), \qquad n \geq 2.$

$$\begin{aligned}
\log^*(2) &= 1, & 2 &= 2 \\
\log^*(2^2) &= 2, & 2^2 &= 4 \\
\log^*(2^{2^2}) &= 3, & 2^{2^2} &= 2^4 = 16 \\
\log^*(2^{2^{2^2}}) &= 4, & 2^{2^{2^2}} &= 2^{16} = 65536 \\
\log^*(2^{2^{2^{2^2}}}) &= 5, & 2^{2^{2^{2^2}}} &= 2^{65536}
\end{aligned}$$

The number of atoms in the observable universe is estimated to be about 10^{80} which is MUCH SMALLER than 2^{65536}!!

In practice, above *union-find* algorithm is linear time.

An implementation of efficient union-find data structure.

```
Make-set($x$)
    $parent[x] = x; \quad rank[x] = 0;$

Union($x, y$)
    Link(Find-set($x$), Find-set($y$));

Link($x, y$)
    if ($rank[x] > rank[y]$)
        $parent[y] = x;$
    else if ($rank[x] < rank[y]$)
        $parent[x] = y;$
    else if ($x \neq y$)
        $parent[y] = x; \quad rank[x] = rank[x] + 1;$

Find-set($x$)
    if $x \neq parent[x]$
        $parent[x] =$ Find-set($parent[x]$)
    return($parent[x]$)
```