Chapter 2 (Solutions)

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2.2-1 (a) Characteristic Polynomial: $\lambda^2+5\lambda+6$

" equation: $\lambda^2+5\lambda+6=0 \longrightarrow$ roots: $\lambda_1=-2$; $\lambda_2=-3$

" modes : e^{-2t}, e^{-3t}

(b) $y_0(t)=5e^{-2t}-3e^{-3t}$.

2.2-2 Characteristic Polynomial: $\lambda^2+4\lambda+4$

" equation: $\lambda^2+4\lambda+4=0 \longrightarrow$ roots : $\lambda_1=\lambda_2=-2$

" modes: e^{-2t}, te^{-2t}

$y_0(t)=(3+2t)e^{-2t}$.

2.2-3 Characteristic Polynomial: $\lambda(\lambda+1)=\lambda^2+\lambda$

" equation: $\lambda(\lambda+1)=0 \longrightarrow$ roots : $\lambda_1=0$; $\lambda_2=-1$

" modes: : $1, e^{-t}$

$y_0(t)=2-e^{-t}$

2.2-4 Characteristic Polynomial: λ^2+9

" equation : $\lambda^2+9=0 \longrightarrow$ roots: $\lambda_1=+j3$; $\lambda_2=-j3$.

" modes : e^{j3}, e^{-j3}

$y_0(t)=2\cos(3t-\frac{\pi}{2})=2\sin(3t)$

2.2-5 Characteristic Polynomial: $\lambda^2+4\lambda+13$

" equation: $\lambda^2+4\lambda+13=0 \longrightarrow$ roots: $\lambda_1=-2+j3$; $\lambda_2=-2-j3$

" modes: $e^{(-2+j3)t}$; $e^{(-2-j3)t}$

$y_0(t)=10e^{-2t}\cos(3t-\pi/3)$

2.2-6 Characteristic Polynomial: $\lambda^2(\lambda+1)=\lambda^3+\lambda^2$

" equation: $\lambda^2(\lambda+1)=0 \longrightarrow$ roots: $\lambda_1=\lambda_2=0$; $\lambda_3=-1$

$y_0(t)=5+2t-e^{-t}$

2.2-7 Characteristic Polynomial: $(\lambda+1)(\lambda^2+5\lambda+6)$

" equation : $(\lambda+1)(\lambda^2+5\lambda+6)=0 \rightarrow$ roots: $\lambda_1=-1$; $\lambda_2=-2$; $\lambda_3=-3$

" modes: e^{-t}, e^{-2t}, e^{-3t}

$y_0(t)=6e^{-t}-7e^{-2t}+3e^{-3t}$.

2.3-1. $y_n(t)=0.5(e^{-t}-e^{-3t})$

$h(t)=[P(D)y_n(t)]u(t)=(2e^{-t}-e^{-3t})u(t)$.

2.3-2 $h(t) = \delta(t) + (e^{-2t} + e^{-3t})u(t)$

2.3-3 $h(t) = -\delta(t) + 2e^{-t}u(t)$

2.3-4 $h(t) = (2+3t)e^{-3t}u(t)$

2.4-1 $A_c = \int_{-\infty}^{+\infty} c(t)\,dt = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\,d\tau\right]dt = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)\,d\tau\right]g(t-\tau)\,dt$

$= A_f \int_{-\infty}^{+\infty} g(t-\tau)\,dt = A_f A_g$.

For example 2.6 : $A_f = 1$; $A_h = 0.5$; $A_y = 1 - 0.5 = 0.5 = A_f A_h$.

For " 2.8 : $A_f = 2$; $A_g = 1.5$; $A_3 = 3 = A_f A_g$.

2.4-2 $f(at) * g(at) = \int_{-\infty}^{+\infty} f(a\tau)\,g[a(t-\tau)]\,d\tau = \frac{1}{a}\int_{-\infty}^{+\infty} f(x)\,g(at-x)\,dx = \frac{1}{a}c(at)$ $a \geq 0$

when $a<0$, the limits of integration become from ∞ to $-\infty$ which is equivalent to the limits from $-\infty$ to ∞ with a negative sign. Hence

$$f(at) * g(at) = \left|\frac{1}{a}\right| c(at)$$

2.4-3 Assume $f(t) * g(t) = c(t) \xrightarrow{\text{using time}}_{\text{Scaling property}} f(-t) * g(-t) = c(-t)$;

now if $f(t)$ and $g(t)$ are both even functions of t then $f(t) = f(-t)$ and $g(t) = g(-t)$. Clearly $c(t) = c(-t)$. Using the same argument if both functions are odd $c(t) = c(-t)$. If one is odd and the other is even $c(t) = -c(-t)$ indicating that $c(t)$ is odd.

2.4-4 $e^{-at}u(t) * e^{-bt}u(t) = \left(\dfrac{e^{-at} - e^{-bt}}{a-b}\right)u(t)$

2.4-5 (i) $u(t) * u(t) = tu(t)$

(ii) $e^{-at}u(t) * e^{-at}u(t) = te^{-at}u(t)$

(iii) $tu(t) * u(t) = \frac{1}{2}t^2u(t)$

2.4-6

(i) $\sin t\, u(t) * u(t) = (1 - \cos t)u(t)$

(ii) $\cos t\, u(t) * u(t) = \sin t\, u(t)$

2.4-7 (a) $y(t) = e^{-t}u(t) * u(t) = (1 - e^{-t})u(t)$

(b) $y(t) = e^{-t}u(t) * e^{-t}u(t) = te^{-t}u(t)$

(c) $y(t)=e^{-t}u(t)*e^{-2t}u(t)=(e^{-t}-e^{-2t})u(t)$

(d) $y(t)=\sin 3t\,u(t)*e^{-t}u(t)=\dfrac{0.9486\,e^{-t}-\cos(3t+18.4^\circ)}{\sqrt{10}}u(t)$.

2.4-8

(a) $y(t)=(2e^{-3t}-e^{-2t})u(t)*u(t)=\left(\frac{1}{6}-\frac{2}{3}e^{-3t}+\frac{1}{2}e^{-2t}\right)u(t)$

(b) $y(t)=(2e^{-3t}-e^{-2t})u(t)*e^{-t}u(t)=(e^{-2t}-e^{-3t})u(t)$.

(c) $y(t)=(2e^{-3t}-e^{-2t})u(t)*e^{-2t}u(t)=[(2-t)e^{-2t}-2e^{-3t}]u(t)$.

2.4-9 $y(t)=(1-2t)e^{-2t}u(t)*u(t)=te^{-2t}u(t)$.

2.4-10 (a) $y(t)=\dfrac{4}{\sqrt{13}}[0.555-e^{-2t}\cos(3t+56.31^\circ)]u(t)$

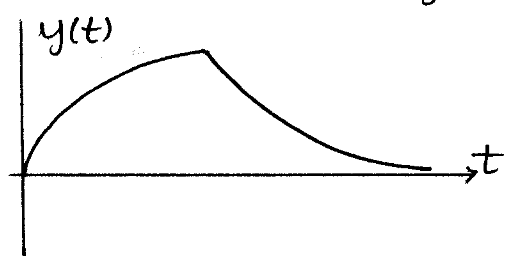
(b) $y(t)=4\left[e^{-t}-\dfrac{1}{\sqrt{10}}e^{-2t}\cos(3t+71.56^\circ)\right]u(t)$.

2.4-11. (a) $y(t)=e^{-t}u(t)*e^{-2t}u(t)=(e^{-t}-e^{-2t})u(t)$.

(b) $y(t)=e^{6}[e^{-t}u(t)*e^{-2t}u(t)]=e^{6}(e^{-t}-e^{-2t})u(t)$.

(c) $y(t)=e^{-6}[e^{-(t-3)}u(t)-e^{-2(t-3)}]u(t-3)$.

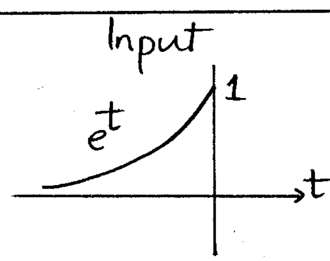
(d) $y(t)=(1-e^{-t})u(t)-[1-e^{-(t-1)}]u(t-1)$.

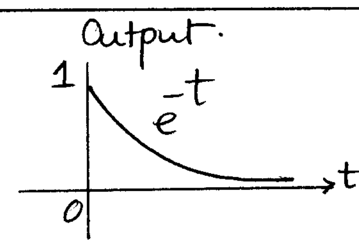


2.4-12

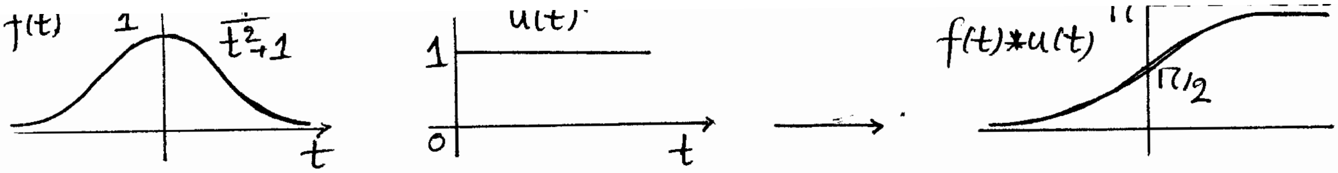
$y(t)=[-\delta(t)+2e^{-t}u(t)]*e^{t}u(-t)$

$=e^{-t}u(t)$





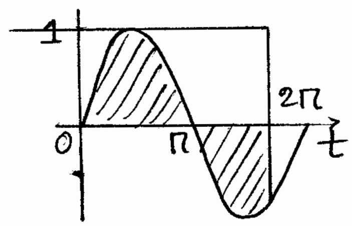
2.4-13 $f(t)*u(t)=\dfrac{1}{t^2+1}*u(t)=\tan^{-1}t+\pi/2$

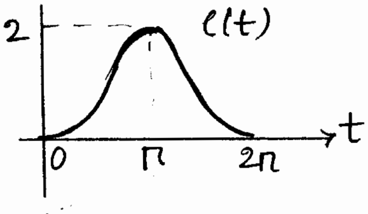


2.4-14

$$c(t) = f(t) * g(t) = 1 - \cos t \qquad 0 \leq t \leq 2\pi$$

$$c(t) = 0 \qquad t \geq 2\pi$$



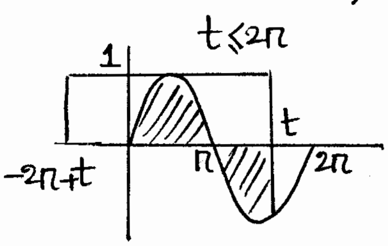


2.4-15

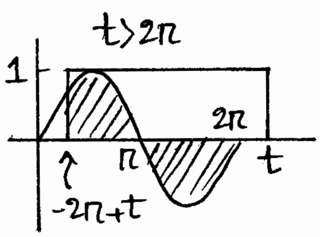
For $0 \leq t \leq 2\pi$: $f(t) * g(t) = 1 - \cos t$

$2\pi \leq t \leq 4\pi$ $\quad f(t) * g(t) = \cos t - 1$

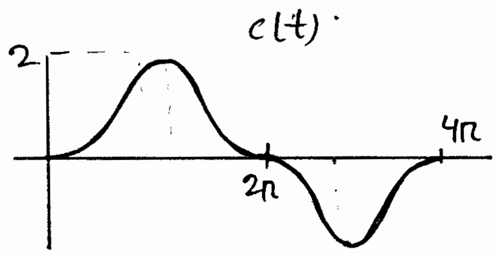
$t > 4\pi$ $\quad f(t) * g(t) = 0$



(a)



(b)

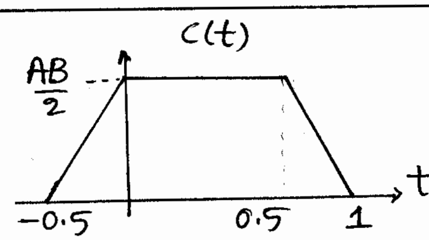


(c)

2.4-16.

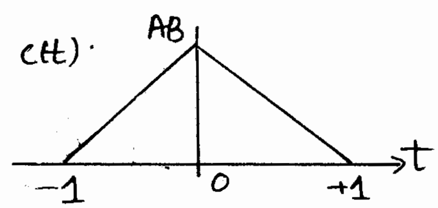
(a)

$$c(t) = \begin{cases} AB & 0 \leq t < 1 \\ AB(2-t) & 1 \leq t \leq 2 \\ AB(t+1) & -1 \leq t \leq 0 \\ 0 & t \geq 2 \text{ or } t \leq -1. \end{cases}$$



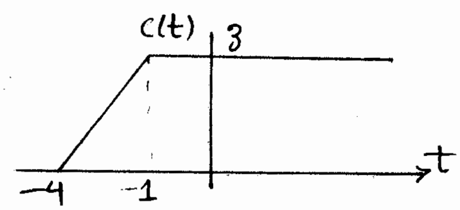
(b)

$$c(t) = \begin{cases} AB(2-t) & 0 \leq t \leq 2 \\ AB(t+2) & -2 \leq t \leq 0 \\ 0 & |t| \geq 2 \end{cases}$$



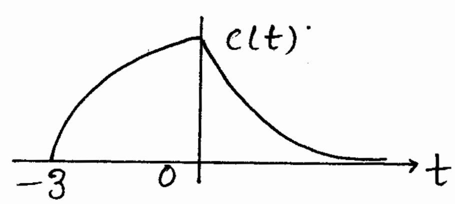
(c)

$$c(t) = \begin{cases} 3 & t > -1 \\ t+4 & -1 \geq t \geq -4 \\ 0 & t \leq -4 \end{cases}$$

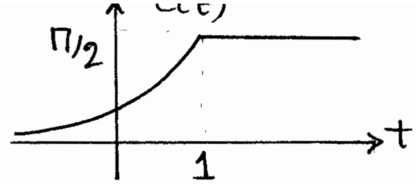


(d)

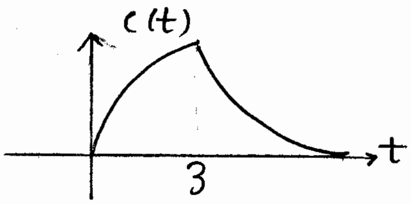
$$c(t) = \begin{cases} 0.95e^{-t} & t \geq 0 \\ 1 - 0.0498e^{-t} & -3 \leq t \leq 0 \\ 0 & t \leq -3 \end{cases}$$



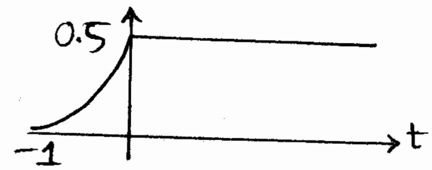
(e) $c(t)=\begin{cases}\tan^{-1}(t-1)+\frac{\pi}{2} & t\leq 1\\ \pi/2 & t\geq 1\end{cases}$



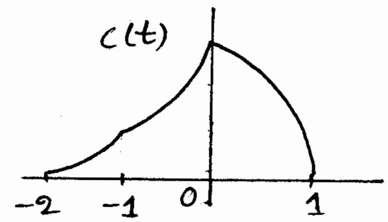
(f) $c(t)=\begin{cases}1-e^{-t} & 0\leq t\leq 3\\ e^{-(t-3)}-e^{-t} & t\geq 3\end{cases}$



(g) $c(t)=\begin{cases}1/2 & t\geq 0\\ \frac{1}{2}(1-t^2) & -1\leq t\leq 0\\ 0 & t\geq 0\end{cases}$



(h) $c(t)=\begin{cases}\frac{1}{3}[e^{-2t}-e^{t-3}] & 0\leq t\leq 1\\ \frac{1}{3}[e^{t}-e^{t-3}] & -1\leq t\leq 0\\ \frac{1}{3}[e^{t}-e^{-2(t+3)}] & -2\leq t\leq -1\\ 0 & t\leq -2\end{cases}$



2.4-17

$$f(t)\longrightarrow y(t)$$
$$f(t-T)\longrightarrow y(t-T)\quad \text{(by time invariance)}$$
$$f(t)-f(t-T)\longrightarrow y(t)-y(t-T)\quad \text{(by linearity)}.$$

Therefore $\lim_{T\to 0}\frac{1}{T}[f(t)-f(t-T)]\longrightarrow \lim_{T\to 0}\frac{1}{T}[y(t)-y(t-T)]$.

The left-hand side is $\dot{f}(t)$ and the right-hand side is $\dot{y}(t)$; therefore

$$\dot{f}(t)\longrightarrow \dot{y}(t)$$

Next we recognize that: $f(t)*u(t)=\int_{-\infty}^{t} f(\tau)u(t-\tau)\,d\tau=\int_{-\infty}^{t} f(\tau)\,d\tau$

This follows from the fact that integration is performed over the range $-\infty<\tau\leq t$ where $\tau\leq t$. Hence $u(t-\tau)=1$. Now the response to $\int_{-\infty}^{t} f(\tau)\,d\tau$ is:

$$[f(t)*u(t)]*h(t)=[f(t)*h(t)]*u(t)=y(t)*u(t).$$

but as shown in Eq. (1), $y(t)*u(t)=\int_{-\infty}^{t} y(\tau)\,d\tau$. Therefore the response to input $\int_{-\infty}^{t} f(\tau)\,d\tau$ is $\int_{-\infty}^{t} y(\tau)\,d\tau$.

2.4-18

$$\dot{f}(t)*g(t)=\lim_{T\to 0}\frac{1}{T}[f(t)-f(t-T)]*g(t)=f(t)*\lim_{T\to 0}\frac{1}{T}[g(t)-g(t-T)]=\lim_{T\to 0}\frac{1}{T}[c(t)-c(t-T)]=\dot{c}(t)$$

Successive application of this procedure yeilds: $f^{(m)}(t)*g^{(n)}(t)=c^{(m+n)}(t)$

2.4-19 $u(t) \longrightarrow g(t)$

$u(t-\tau) \longrightarrow g(t-\tau)$

The input $f(t)$ is made up of step components. The step component at τ has a height Δf which can be expressed as:

$$\Delta f = \frac{\Delta f}{\Delta \tau}\Delta\tau = \dot{f}(\tau)\Delta\tau.$$

The step component at $n\Delta\tau$ has a height $\dot{f}(n\Delta\tau)\Delta\tau$ and can be expressed as $[\dot{f}(n\Delta\tau)\Delta\tau]u(t-n\Delta\tau)$. Its response $\Delta y(t)$ is:

$$\Delta y(t) = [\dot{f}(n\Delta\tau)\Delta\tau]\,g(t-n\Delta\tau)$$

The total response due to all components is $y(t) = \lim_{\Delta\tau\to 0}\sum_{n=-\infty}^{\infty}\dot{f}(n\Delta\tau)g(t-n\Delta\tau)\Delta\tau$

$$= \int_{-\infty}^{+\infty}\dot{f}(\tau)\,g(t-\tau)\,d\tau = \dot{f}(\tau) * g(\tau)$$

2.4-20
An element of length $\Delta\tau$ at point $n\Delta\tau$ has a charge $f(n\Delta\tau)\Delta\tau$. The electric field due to this charge at point x is: $\Delta E = \dfrac{f(n\Delta\tau)\Delta\tau}{4\pi\epsilon(x-n\Delta\tau)^2} \longrightarrow E = \lim_{\Delta\tau\to 0}\sum_{n=-\infty}^{\infty}\dfrac{f(n\Delta\tau)\Delta\tau}{4\pi\epsilon(x-n\Delta\tau)^2}$

$$= \int_{-\infty}^{+\infty}\frac{f(\tau)}{4\pi\epsilon(x-\tau)^2}\,d\tau = f(x) * \frac{1}{4\pi\epsilon x}$$

2.4-21 $H(s) = \int_{-\infty}^{+\infty}\delta(\tau-T)e^{-s\tau}\,d\tau = e^{-sT}$ [Impulse response is: $h(t) = \delta(t-T)$

The same result can be obtained using Eq. (2.49). Let the input to an ideal delay of T seconds be an everlasting exponetial e^{st}. The output is $e^{s(t-T)}$. Hence according to equation (2.49) $H(s) = e^{s(t-T)}/e^{st} = e^{-sT}$

2.5-1. $\lambda^2+7\lambda+12 = (\lambda+3)(\lambda+4) \xrightarrow{\text{natural response}} y_n(t) = K_1e^{-3t} + K_2e^{-4t}$.

(a) $y(t) = \frac{1}{3}e^{-3t} - \frac{1}{2}e^{-4t} + \frac{1}{6}$ $t \geqslant 0$

(b) $y(t) = \frac{1}{2}e^{-3t} - \frac{2}{3}e^{-4t} + \frac{1}{6}e^{-t}$ $t \geqslant 0$

(c) $y(t) = e^{-3t} - e^{-4t}$ $t \geqslant 0$

2.5-2 $y(t) = 0.427\,e^{-3t}\cos(4t - 106.3^\circ) + 3/25$ $t \geqslant 0$

2.5-3 (a) $y(t) = \left(\frac{17}{4} + \frac{15}{2}t\right)e^{-2t} - 2e^{-3t}$ $t \geqslant 0$

(b) $y(t) = \left(\frac{9}{4} + \frac{19}{2}t\right)e^{-2t}$ $t \geqslant 0$

2.5–4 $y(t) = \frac{9}{4} - \frac{1}{4}e^{-2t} + \frac{1}{2}t \quad t \geq 0$

2.5–5 $y(t) = 2e^{-3t} - 2e^{-4t} - te^{-3t}$
$= (2-t)e^{-3t} - 2e^{-4t} \quad t \geq 0$

2.6–1
(a) The system is asymptotically stable
(b) The system is marginally stable.
(c) The system is unstable.
(d) The system is unstable.

2.6–2
(a) The system is asymptotically stable
(b) The system is marginally stable.
(c) The system is unstable
(d) The system is marginally stable.

2.6–3 (a) Because $u(t) = e^{0t}u(t)$, the characteristic root is 0.
(b). The system is marginally stable.
(c) $\int_0^\infty h(t)\,dt = \infty$
(d) Ideal integrator.

2.6–4. Assume that a system exist that violates Eq (2.57) and yet produce a bounded output for every bounded input. The response at $t = t_1$ is: $y(t_1) = \int_0^\infty h(\tau) f(t_1 - \tau)\,d\tau$
Consider a bounded input $f(t)$ such that at some instant t_1:

$$f(t_1 - \tau) = \begin{cases} 1 & \text{if } h(\tau) > 0 \\ -1 & \text{if } h(\tau) < 0 \end{cases}$$

In this case $h(\tau) f(t_1 - \tau) = |h(\tau)|$ and $y(t_1) = \int_0^\infty |h(\tau)|\,d\tau = \infty$
which violates the assumption.

2.7–1 (a) Time constant (rise time): $T_h = 10^{-5}$; (b) bandwidth $B = \frac{1}{T_h} = 10^5\ Hz$ KHz
The channel can transmit signals of BW=15

2.7–2 $T_h = 1/B = 0.1\ ms$; Maximum pulse rate $= \frac{1}{0.6 \times 10^{-3}} \simeq 1667$ pulses/Sec.

2.7–3 (a) $T_r = T_h = -1/\lambda = 10^{-4}$
(b) $F_c = 1/T_h = 1/T_r = 10^4$
(c) Pulse transmission rate is $F_c = 10^4$ pulses/Sec.