# Chapter 2 (Complete solution for selected Problems)

Prepared by: Abbas Ebrahimi-Moghadam (ebrahia@mcmaster.c

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**(2.2-7)**

$$\begin{cases} (D+1)(D^2+5D+6)\,y(t) = Df(t) \\ y_0(0)=2\,,\ \dot{y}_0(0)=-1\,,\ \ddot{y}_0(0)=5 \end{cases}$$

The characteristic polynomial is $(\lambda+1)(\lambda^2+5\lambda+6)$ So the characteristic equation is: $(\lambda+1)(\lambda^2+5\lambda+6)=(\lambda+1)(\lambda+2)(\lambda+3)=0$. The characteristic roots are: $-1$, $-2$ and $-3$. The characteristic modes are $e^{-t}$, $e^{-2t}$, and $e^{-3t}$

Therefore $\quad y_0(t)= c_1e^{-t}+c_2e^{-2t}+c_3e^{-3t}$

and $\quad \dot{y}_0(t)=-c_1e^{-t}-2c_2e^{-2t}-3c_3e^{-3t}$

$\quad \ddot{y}_0(t)= c_1e^{-t}+4c_2e^{-2t}+9c_3e^{-3t}$

setting $t=0$ and substituting initial conditions yields

$$\begin{cases} c_1+c_2+c_3=2 \\ -c_1-2c_2-3c_3=-1 \\ c_1+4c_2+9c_3=5 \end{cases} \Rightarrow c_1=6\,,\ c_2=-7\,,\ c_3=3$$

Therefore: $\quad y_0(t)=6e^{-t}-7e^{-2t}+3e^{-3t}$

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**(2.3-4)** The characteristic equation is $(\lambda^2+6\lambda+9)=(\lambda+3)^2=0$

Therefore $\quad y_n(t)=(c_1+c_2t)e^{-3t}$

$\quad \dot{y}_n(t)=[-3(c_1+c_2t)+c_2]e^{-3t}$

setting $t=0$ and substituting $y_n(0)=1$, $\dot{y}_n(0)=1$ we have,

$$\begin{cases} c_1=0 \\ -3c_1+c_2=1 \end{cases} \Rightarrow c_1=0\,,\ c_2=1$$

$$\Rightarrow y_n(t)= te^{-3t}$$

using (2.19): $\quad h(t)=b_n\delta(t)+[P(D)y_n(t)]u(t)$

$$= [2\dot{y}_n(t)+9y_n(t)]u(t)=(2+3t)e^{-3t}u(t)$$

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**(2.4.5)**

(a) $\quad u(t)*u(t)=\int_{-\infty}^{\infty}u(\tau)u(t-\tau)\,d\tau=\int_0^{t}\overset{1}{u(\tau)}\,\overset{1}{u(t-\tau)}\,d\tau=\tau\Big|_0^t=t \quad$ for $t\geq 0$

$= 0 \quad$ for $t<0$

Therefore $\quad u(t)*u(t)=t\,u(t)$

(b) $$e^{-at}u(t) * e^{-at}u(t) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)\cdot e^{-a(t-\tau)}u(t-\tau)\,d\tau$$

$$= \int_0^t e^{-a\tau} e^{-a(t-\tau)}\,d\tau = e^{-at}\int_0^t d\tau = te^{-at} \quad t \geq 0$$

$$= 0 \quad t < 0$$

Then : $e^{-at}u(t) * e^{-at}u(t) = te^{-at}u(t)$

(c) $$t\,u(t) * u(t) = \int_{-\infty}^{\infty} (\tau u(\tau))\cdot(u(t-\tau))\,d\tau\,.$$

$$= \int_0^t \tau\,d\tau = \frac{t^2}{2} \quad t \geq 0$$

$$= 0 \quad t < 0$$

$$\Rightarrow \quad t\,u(t) * u(t) = \frac{1}{2}t^2 u(t)$$

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**2.4-10** $\quad h(t) = 4e^{-2t}\cos 3t\,u(t)$

(a) For $y(t) = 4e^{-2t}\cos 3t\,u(t) * u(t)$ we use pair 12 from table 2.1 with $\alpha=2$, $\beta=3$, $\theta=0$, $\lambda=0$, Therefore $\phi = \tan^{-1}[\frac{-3}{2}] = -56.31^\circ$

and $$y(t) = 4\left[\frac{\cos(56.31^\circ) - e^{-2t}\cos(3t+56.31^\circ)}{\sqrt{4+9}}\right]$$

$$= \frac{4}{\sqrt{13}}\left[0.555 - e^{-2t}\cos(3t+56.31^\circ)\right]u(t)$$

(b) For $y(t) = 4e^{-2t}\cos 3t\,u(t) * e^{-t}u(t)$, we use Pair 12 from table 2 with $\alpha=2$, $\beta=3$, $\theta=0$, $\lambda=-1$. Therefore $\phi = \tan^{-1}[\frac{-3}{1}] = -71.56^\circ$

and $$y(t) = 4\left[\frac{\cos(71.56^\circ)e^{-t} - e^{-2t}\cos(3t+71.56^\circ)}{\sqrt{10}}\right]u(t)$$

$$= \frac{4}{\sqrt{10}}\left[0.316e^{-t} - e^{-2t}\cos(3t+71.56^\circ)\right]u(t)$$

$$= 4\left[e^{-t} - \frac{1}{\sqrt{10}}e^{-2t}\cos(3t+71.56^\circ)\right]u(t)$$

**2.4-12**

$$y(t) = \left[-\delta(t) + 2e^{-t}u(t)\right] * e^{t}u(-t)$$

distributive property $$= -\delta(t) * e^{t}u(-t) + 2e^{-t}u(t) * e^{t}u(-t)$$

Convolution with impulse $$= -e^{t}u(-t) + \underbrace{2e^{-t}u(t) * e^{t}u(-t)}_{y_1(t)}$$

$$y_1(t) = 2\int_{-\infty}^{\infty} e^{-\tau}u(\tau)\, e^{t-\tau}u(\tau-t)\,d\tau$$

$$= \begin{cases} 2\int_{0}^{\infty} e^{-\tau} e^{t-\tau}\,d\tau & \text{for } t\le 0 \\ 2\int_{t}^{\infty} e^{-\tau} e^{t-\tau}\,d\tau & \text{for } t>0 \end{cases}$$

$$= \begin{cases} e^{t} & t\le 0 \\ e^{-t} & t>0 \end{cases} = e^{-t}u(t) + e^{t}u(-t)$$

$$\Rightarrow\ y(t) = -e^{t}u(-t) + e^{-t}u(t) + e^{t}u(-t) = e^{-t}u(t)$$

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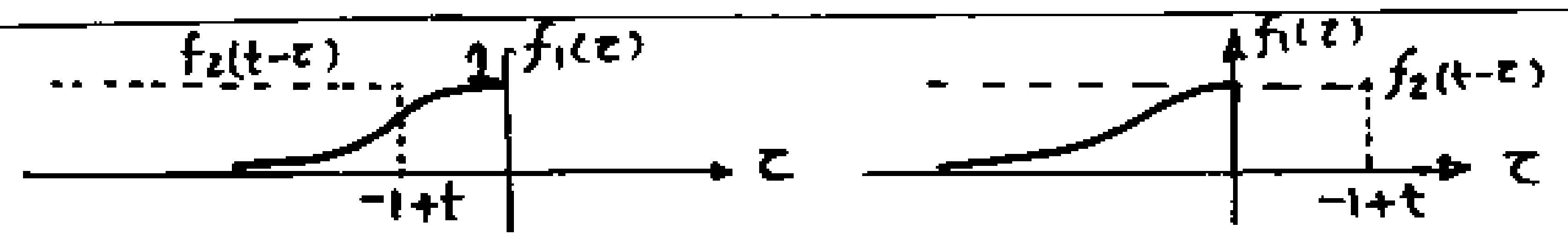
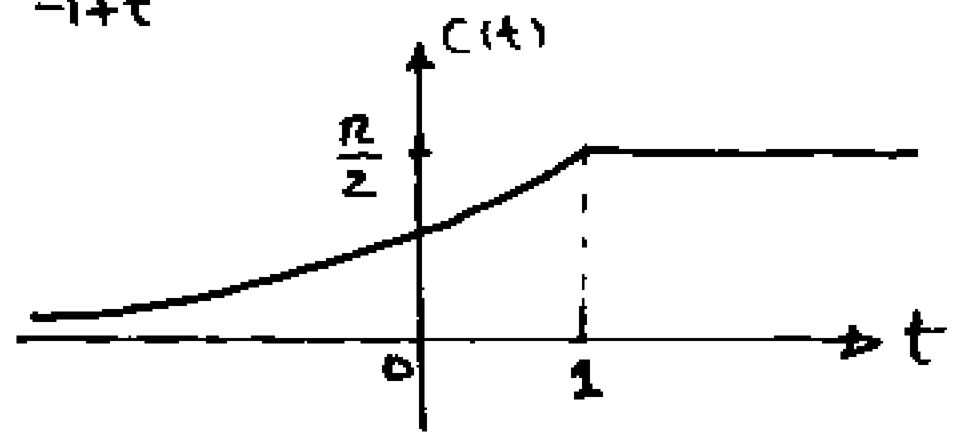
**2.4-16**



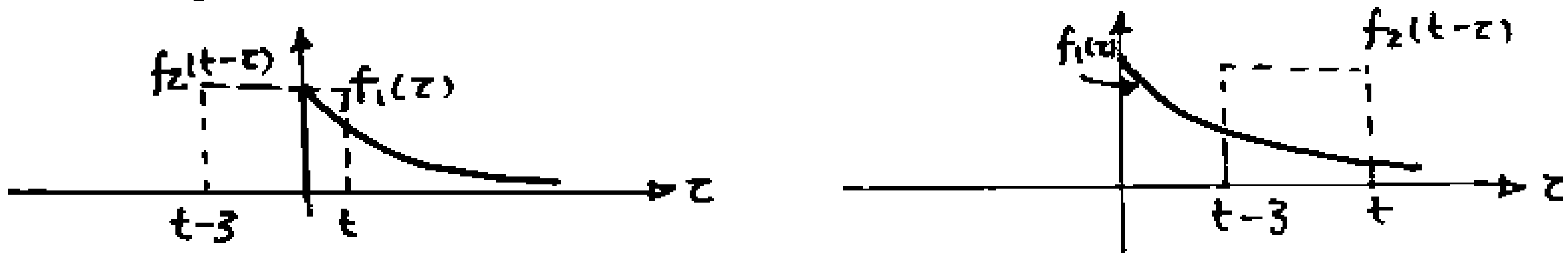

(e)

$$c(t) = \int_{-\infty}^{-1+t} \frac{1}{1+\tau^2}\,d\tau = tg^{-1}(t-1) + \frac{\pi}{2} \qquad t\le 1$$

$$c(t) = \int_{-\infty}^{0} \frac{1}{1+\tau^2}\,d\tau = tg^{-1}\tau\,\Big|_{-\infty}^{0} = \frac{\pi}{2} \qquad t \ge 0$$




(f)




$$c(t) = 0 \qquad t\le 0$$

$$c(t) = \int_{0}^{t} e^{-\tau}\,d\tau = 1 - e^{-t} \qquad 0\le t\le 3$$

$$c(t) = \int_{t-3}^{t} e^{-\tau}\,d\tau = e^{-(t-3)} - e^{-t} \qquad t \ge 3$$

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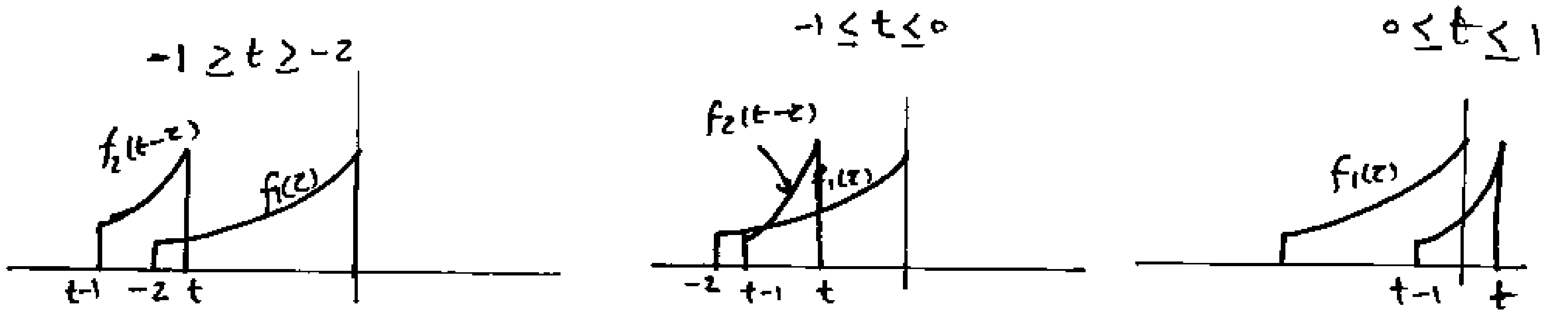
(h) 

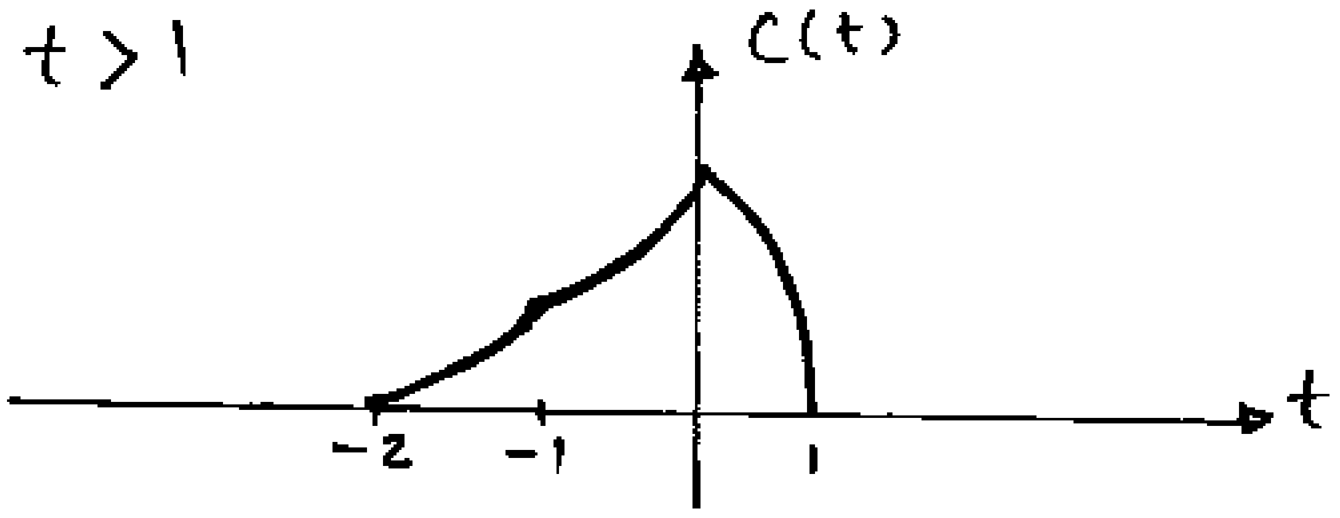

$c(t) = 0 \qquad t \leq -2$

$$c(t) = \int_{-2}^{t} e^{\tau} e^{-2(t-\tau)} d\tau = e^{-2t}\int_{-2}^{t} e^{3\tau} d\tau = \frac{1}{3}\left[e^{t} - e^{-2(t+3)}\right] \qquad -1 \geq t \geq -2$$

$$c(t) = \int_{t-1}^{t} e^{\tau} e^{-2(t-\tau)} d\tau = e^{-2t}\int_{t-1}^{t} e^{3\tau} d\tau = \frac{1}{3}\left[e^{t} - e^{t-3}\right] \qquad -1 \leq t \leq 0$$

$$c(t) = \int_{t-1}^{0} e^{\tau} e^{-2(t-\tau)} d\tau = e^{-2t}\int_{t-1}^{0} e^{3\tau} d\tau = \frac{1}{3}\left[e^{-2t} - e^{t-3}\right] \qquad 0 \leq t \leq 1$$

$c(t) = 0 \qquad t > 1$

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2.5-4 $\quad (D^2+2D)y(t) = (D+1)f(t) \quad , y(0^+) = 2, \; \dot{y}(0^+) = 1 \quad , f(t) = u(t)$

characteristic equation: $(\lambda^2 + 2\lambda) = \lambda(\lambda+2) \Rightarrow$ characteristic roots: $0, -2$

$\Rightarrow \; y_n(t) = K_1 + K_2 e^{-2t}$

In this case $f(t) = u(t)$. The input itself is a characteristic mode. Therefore

$y_\phi(t) = \beta t$

But $y_\phi(t)$ satisfied the system equation

$$(D^2+2D)y_\phi(t) = (D+1)y(t) = \ddot{y}_\phi(t) + 2\dot{y}_\phi(t) = \dot{f}(t) + f(t)$$

substituting $f(t) = u(t)$ and $y_\phi(t) = \beta t$ we obtain

$$0 + 2\beta = 0 + 1 \Rightarrow \beta = 1/2 \quad \Rightarrow y_\phi(t) = \frac{1}{2}t$$

$$y(t) = K_1 + K_2 e^{-2t} + \frac{1}{2}t$$

$$\dot{y}(t) = -2K_2 e^{-2t} + \frac{1}{2}$$

setting $t=0$ and substituting initial conditions yields:

$$\begin{cases} K_1 + K_2 = 2 \\ -2K_1 + \frac{1}{2} = 1 \end{cases} \Rightarrow K_1 = 9/4, \; K_2 = -1/4$$

$$\Rightarrow \quad y(t) = 9/4 + 1/4 e^{-2t} + \frac{1}{2}t \qquad t \geq 0$$

2.7–2

$$T_h = \frac{1}{B} = \frac{1}{10^4} = 10^{-4} = 0.1 \text{ ms}$$

The received pulse width = $(0.5+0.1) = 0.6$ ms. Each pulse takes up 0.6 ms interval. The maximum pulse rate (to avoid interference between successive pulses) is

$$\frac{1}{0.6 \times 10^{-3}} \simeq 1667 \text{ pulses/sec}$$