Chapter 3 Answers Prepared by: Abbas Ebrahimi-Moghadam
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3.1-1) $e = (f - c\underline{x}) \Rightarrow |e|^2 = (f - c\underline{x})(f - c\underline{x})$
$= |f|^2 + c^2|x|^2 - 2c f \cdot \underline{x}$

To minimize the error $\frac{d\,|e|^2}{dc} = 0 \Rightarrow 2c|\underline{x}|^2 - 2f \cdot \underline{x} = 0$
$\Rightarrow c = \frac{f \cdot \underline{x}}{|\underline{x}|^2}$

3.1-2) a) $c = \frac{1}{E_x}\int_0^1 f(t)x(t)\,dt = \frac{1}{1}\int_0^1 t\,dt = 0.5$

b) $f(t) \approx 0.5\,x(t)$ Then error $e(t) = t - 0.5$ over $(0 \le t \le 1)$
$= 0$ outside this interval

$\Rightarrow E_e = \int_0^1 (t - 0.5)^2 dt = \frac{1}{12}$

On the other hand we have : $E_f = \int_0^1 f^2(t)\,dt = \int_0^1 t^2\,dt = 1/3$

If error signal is orthogonal to the signal estimation we have
$E_f = c^2 E_x + E_e$
By inspection we can see that this is right and e is orthogonal to the component $c\underline{x}$

3.1-3) $c = \frac{1}{E_f}\int_0^1 x(t)f(t)\,dt = \frac{1}{1/3}\int_0^1 t\,dt = 1.5$

Thus, $x(t) \approx 1.5 f(t)$ and $e(t) = x(t) - 1.5 f(t) = \begin{cases} 1 - 1.5t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

$E_e = 1/4$

3.1-4) a) $E_x = 0.5$, $c = -1/\pi$, b) $f(t) \approx -(1/\pi)x(t)$, $e(t) = \begin{cases} t + (1/\pi)\sin t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

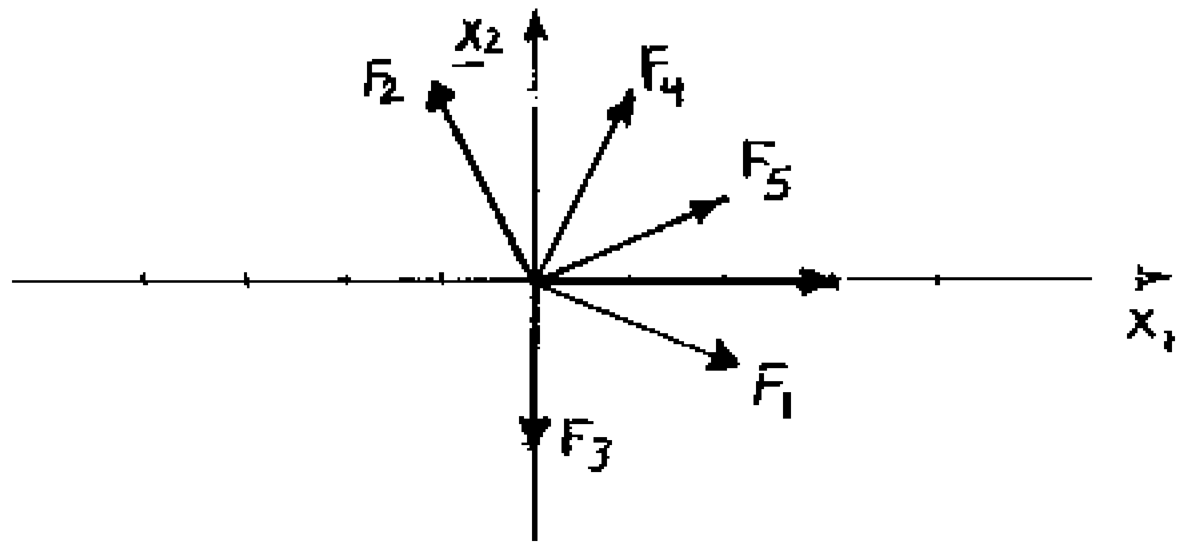
$E_f = 1/3$, $E_e = \frac{1}{3} - \frac{1}{2\pi^2}$

3.1-5) Solved in detailed solution of selected problems.

3.2-1) (1) 0 (2) −1 (3) 0 (4) $1.414/\pi$

signals $x(t)$ and $f_2(t)$ provide the maximum protection against noise.

3.3-1)



$f_1(2,-1)$, $f_2(-1,2)$, $f_3(0,-2)$, $f_4(1,2)$, $f_5(2,1)$, $f_6(3,0)$

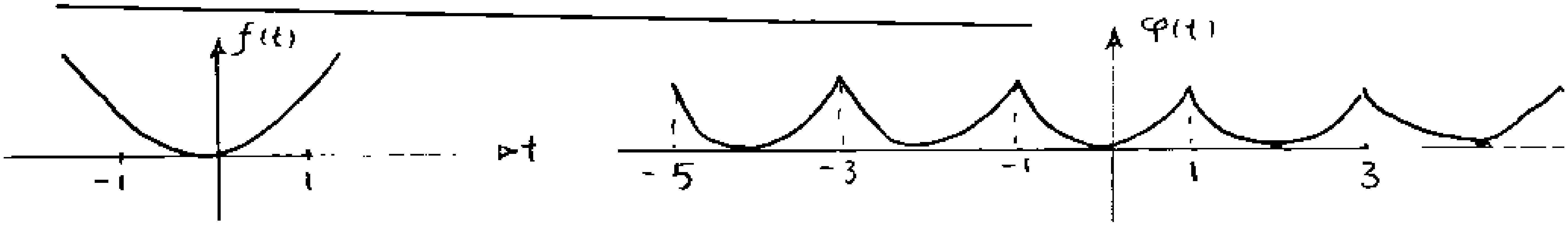
We see that pairs (f_3, f_6), (f_1, f_4), (f_2, f_5) are orthogonal.

$$f_3 \cdot f_6 = (0\times 3) + (-2\times 0) = 0 = \int_{-\infty}^{\infty} (-x_2(t))(3x_1(t))\,dt$$

$$f_1 \cdot f_4 = (2\times 1) + (-1\times 2) = 0 = \int_{-\infty}^{\infty} (2x_1(t) - x_2(t))(x_1(t) + 2x_2(t))\,dt$$

$$f_2 \cdot f_5 = (-1\times 2) + (2\times 1) = 0 = \int_{-\infty}^{\infty} (-x_1(t) + 2x_2(t))(2x_1(t) + x_2(t))\,dt$$

3.4-1)

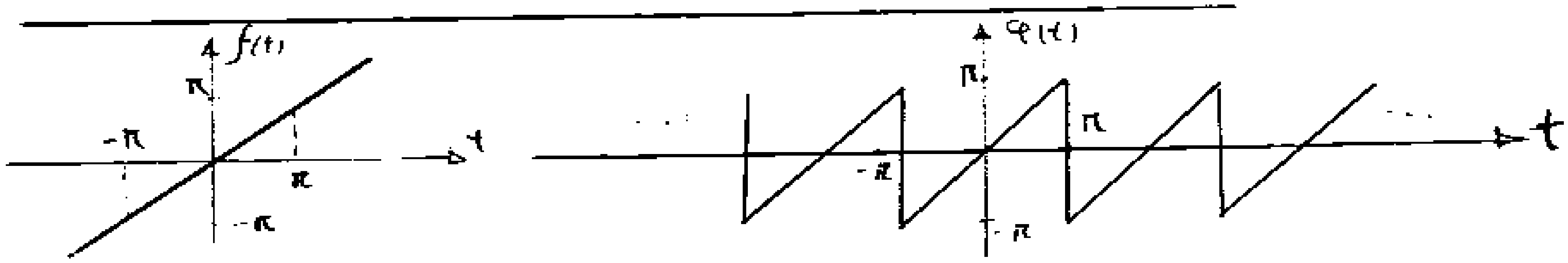


$$\varphi(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi t + b_n \sin n\pi t \qquad -1 \le t \le 1$$

$$a_0 = 1/3, \quad a_n = \frac{4(-1)^n}{\pi^2 n^2}, \quad b_n = 0$$

Therefore ; $$\varphi(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t$$

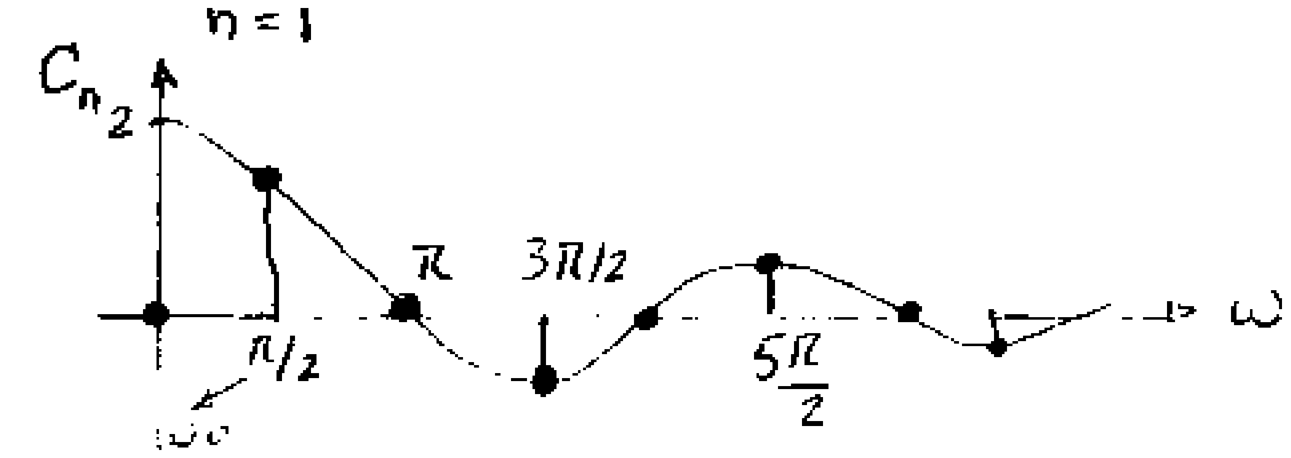
3.4-2)



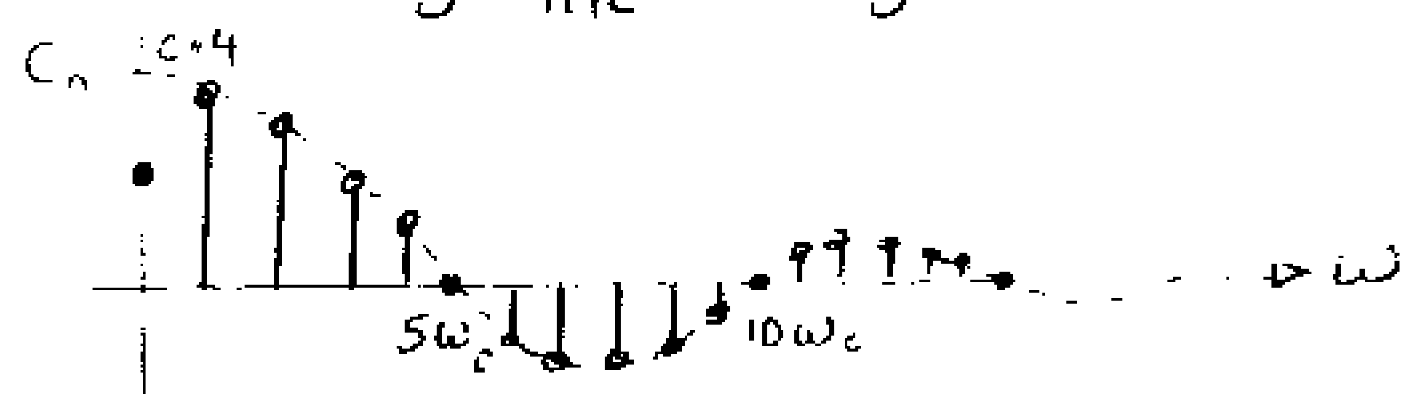
$$\varphi(t) = 2(-1)^{n+1} \sum_{n=1}^{\infty} \frac{1}{n} \sin nt$$

3.4-3)

(a) $f(t)=\sum_{n=1}^{\infty}\frac{4}{n\pi}\sin\frac{n\pi}{2}\cos\left(\frac{n\pi}{2}t\right)$ signal is even

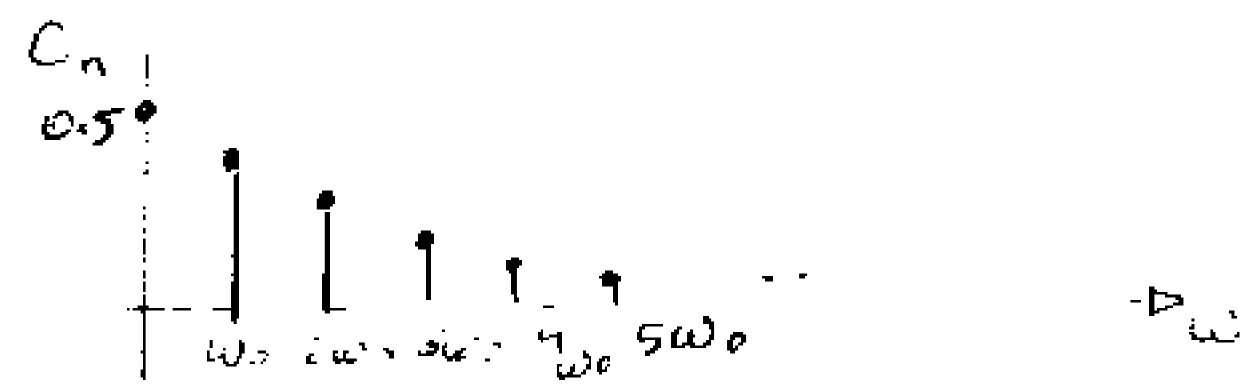


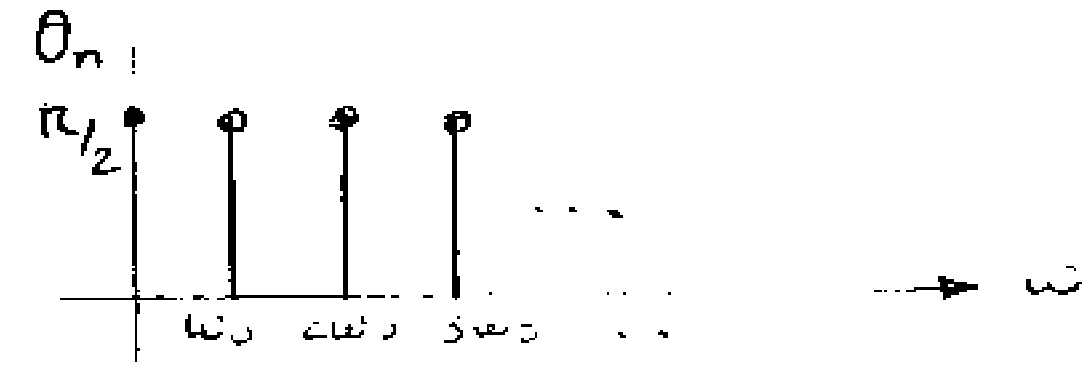
(b) $f(t)=\frac{1}{5}\frac{2}{n\pi}\sin\left(\frac{n\pi}{5}\right)\cos\left(\frac{n}{5}t\right)$ signal is even



(c) $f(t)=0.5+\frac{1}{\pi}\left[\cos\left(t+\frac{\pi}{2}\right)+\frac{1}{2}\cos\left(2t+\frac{\pi}{2}\right)+\frac{1}{3}\cos\left(3t+\frac{\pi}{2}\right)+\cdots\right]$

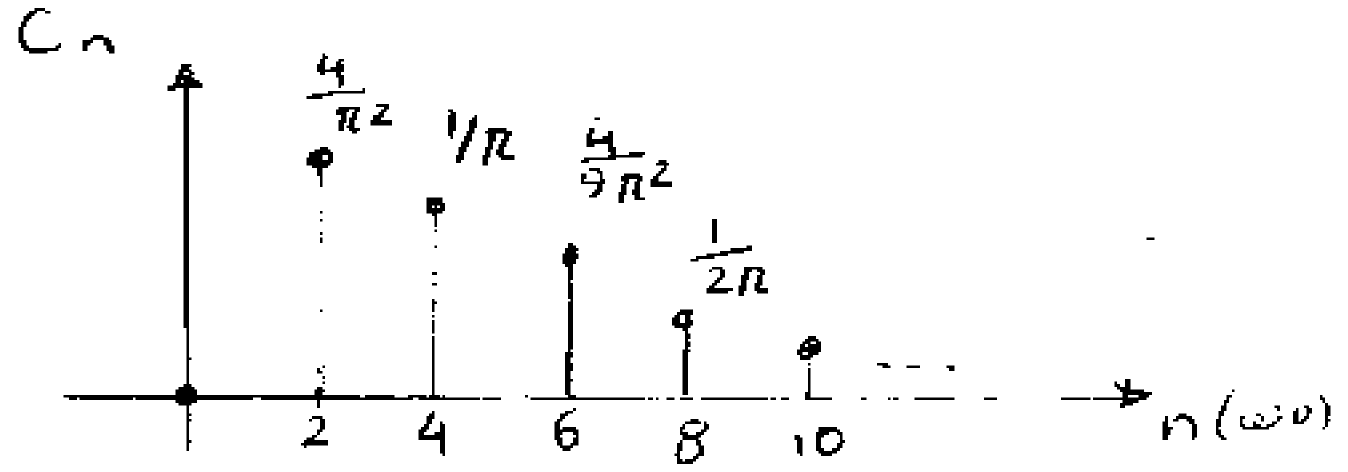
$f(t)-0.5$ is odd so Cosines term vanish

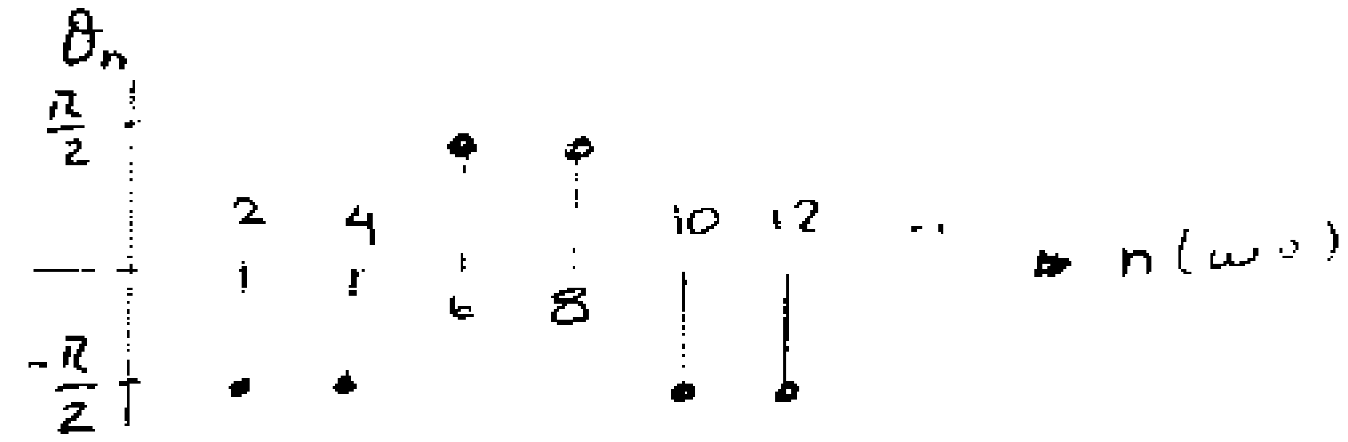




(d) $f(t)=\frac{4}{\pi^2}\sin 2t+\frac{1}{\pi}\sin 4t-\frac{4}{9\pi^2}\sin 6t-\frac{1}{2\pi}\sin 8t+\cdots$ odd signal

$=\frac{4}{\pi^2}\cos\left(2t-\frac{\pi}{2}\right)+\frac{1}{\pi}\cos\left(4t-\pi/2\right)+\frac{4}{9\pi^2}\cos\left(6t+\frac{\pi}{2}\right)+\frac{1}{\pi}\cos\left(8t+\frac{\pi}{2}\right)+\cdots$



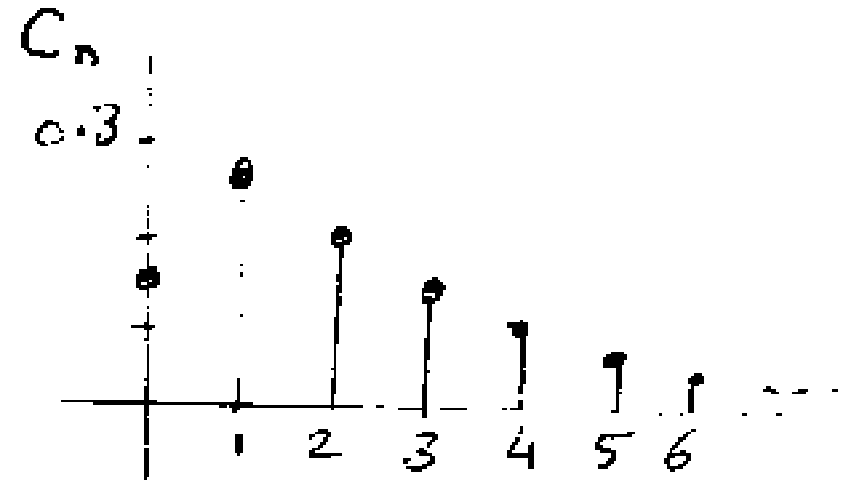


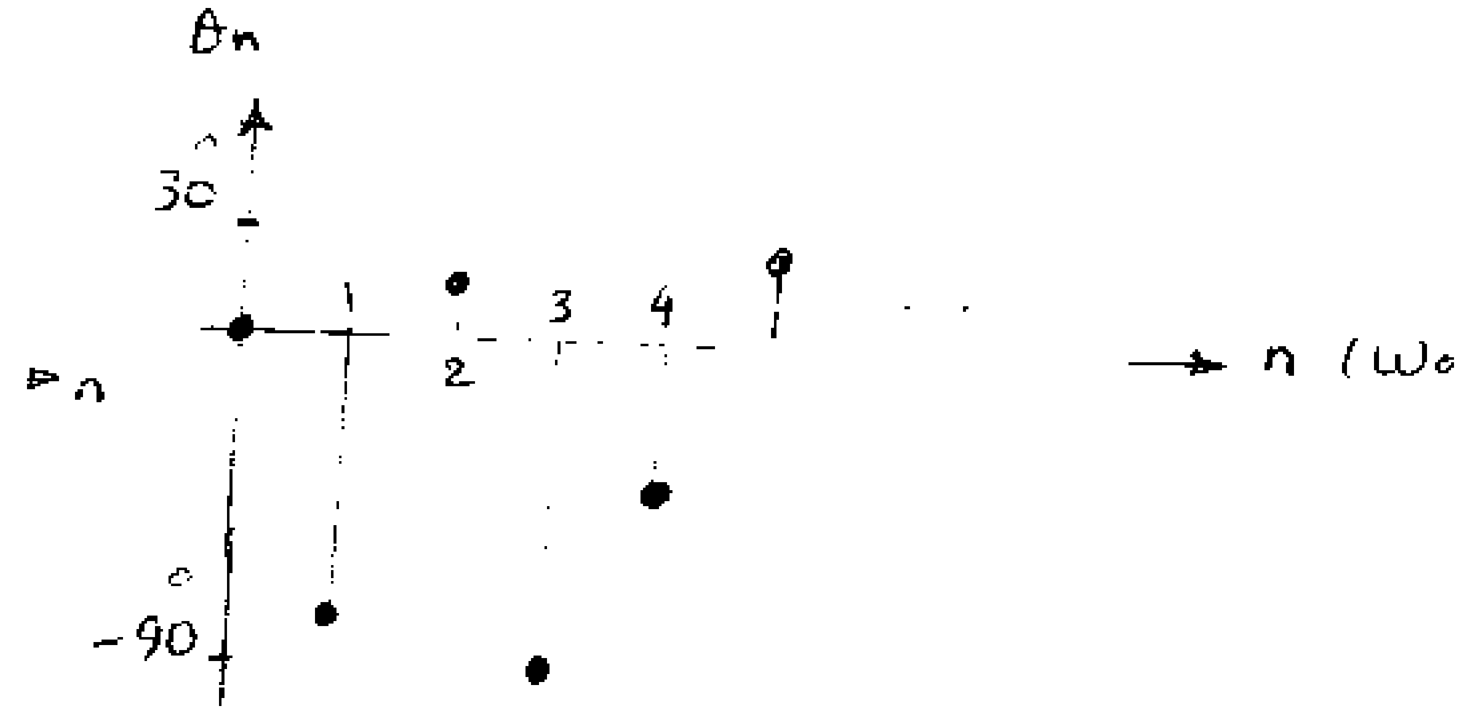
(e)

$$f(t)=\frac{1}{6}+\sum_{n=1}^{\infty}\left(\frac{3}{2\pi^2n^2}\left[\cos\frac{2\pi n}{3}+\frac{2\pi n}{3}\sin\frac{2\pi n}{3}-1\right)\cos\frac{2\pi n t}{3}$$

$$+\left(\frac{3}{2\pi^2n^2}\left[\sin\frac{2\pi n}{3}-\frac{2\pi n}{3}\cos\frac{2\pi n}{3}\right]\sin\frac{2\pi n t}{3}\right.$$

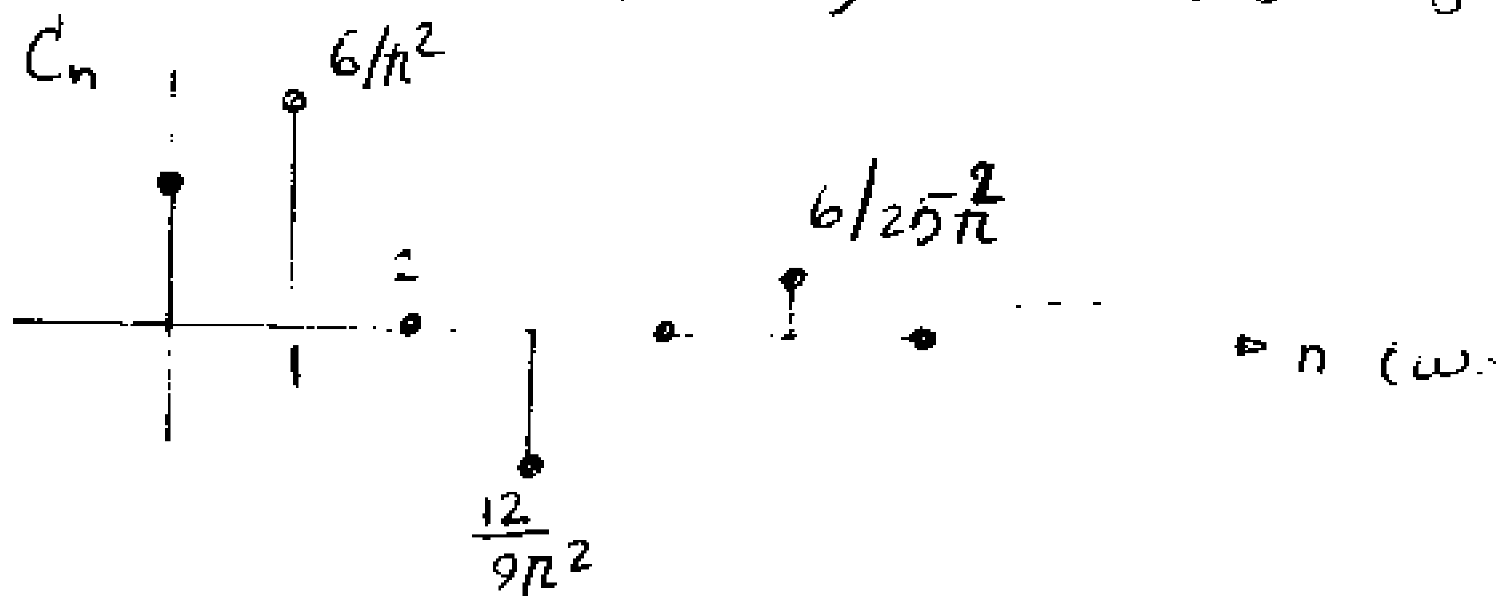
$$C_0=\frac{1}{6},\quad C_n=\frac{3}{2\pi^2n^2}\left[\sqrt{2+\frac{4\pi^2}{9}-9\cos\frac{2\pi n}{3}-\frac{4\pi n}{3}}\right],\quad \theta_n=\tan^{-1}\left(\frac{\frac{2\pi}{3}\cos\frac{2\pi n}{3}-\sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3}+\frac{2\pi n}{3}\sin\frac{2\pi n}{3}-1}\right)$$





(f)

$$f(t) = 0.5 + \frac{6}{\pi^2}\left(\cos\frac{\pi}{3}t - \frac{2}{9}\cos\pi t + \frac{1}{25}\cos\frac{5\pi}{3} + \frac{1}{49}\cos\frac{7\pi}{3} + \cdots\right)$$



3.4-4) a) $$f(t) = 0.504 + 0.504\sum_{n=1}^{\infty}\frac{2}{\sqrt{1+16n^2}}\cos(2nt + \tan^{-1}4n)$$

b) With replacing t by $-t$ Fourier series is identical to (3.56a)

(c) $$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$\Rightarrow f(-t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(-n\omega_0 t + \theta_n)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \theta_n)$$

So time inversion does not affect the amplitude spectrum and just changes the sign of phase spectrum.

3.4-5)

(a) $$f(t) = 0.504 + \sum_{n=1}^{\infty} 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right)\cos(4nt - \tan^{-1}4n)$$

(b) This Fourier series is identical to that in Eq. (3.56a) with replacing t by $2t$

(C) If $f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$

then $f(at) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n(a\omega_0)t + \theta_n)$

3.4-6) a) $f(t) = \frac{8A}{\pi^2}\left[\cos \pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \frac{1}{49}\cos 7\pi t + \cdots\right]$

b) This Fourier series is identical to that in eq. (3.63) with t replaced by $t+0.5$

c) If $f(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n)$

then $f(t+T) = C_0 + \sum C_n \cos[n\omega_0(t+T) + \theta_n]$

$= C_0 + \sum C_n \cos[n\omega_0 t + (n\omega_0 T + \theta_n)]$

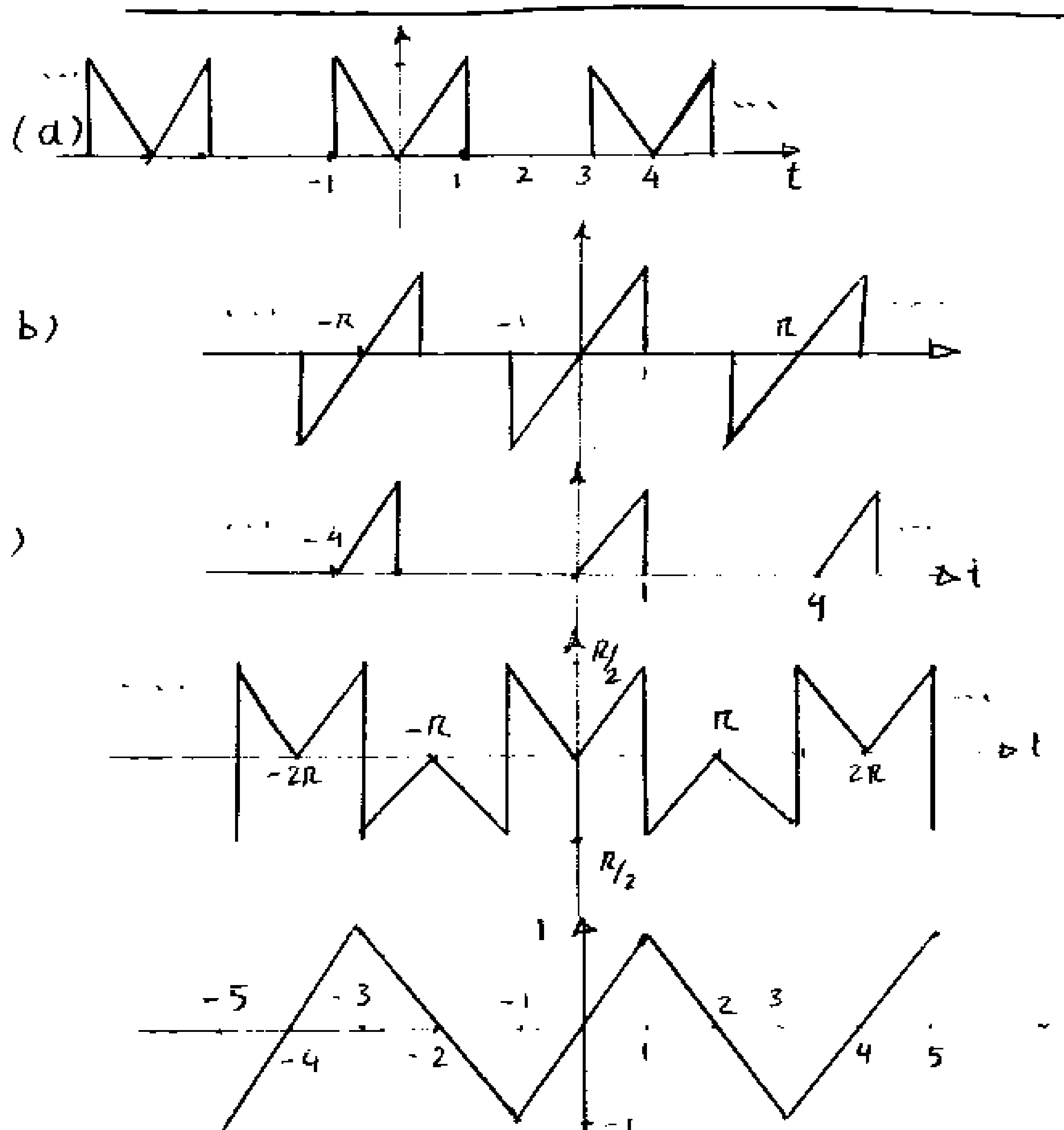
3.4-7) a) $$f(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n \cos \frac{n\pi}{4} t + b_n \sin \frac{n\pi}{4} t$$

where $$a_n = \begin{cases} \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} - 1\right) & n = 1, 5, 9, 13, \cdots \\ \frac{-4}{n^2\pi^2}\left(\frac{n\pi}{2} + 1\right) & n = 3, 7, 11, 15, \cdots \end{cases}$$

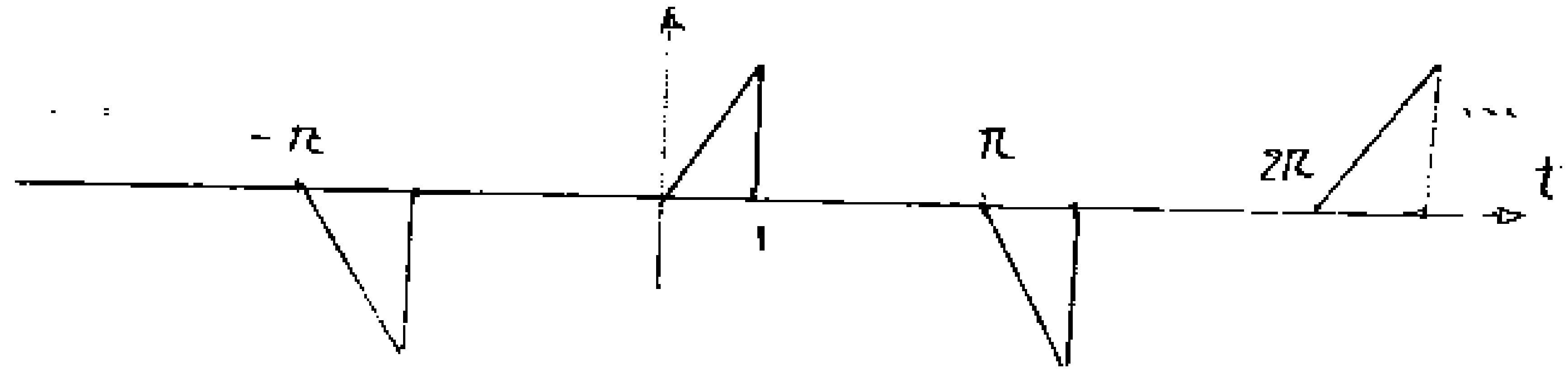
$$b_n = \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \qquad n = 1, 3, 5, \cdots$$

b) $$f(t) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{0.0465}{n^2 + 0.01} \cos nt + \frac{1.461n}{n^2+0.01} \sin nt$$

3.4-8)



(f)

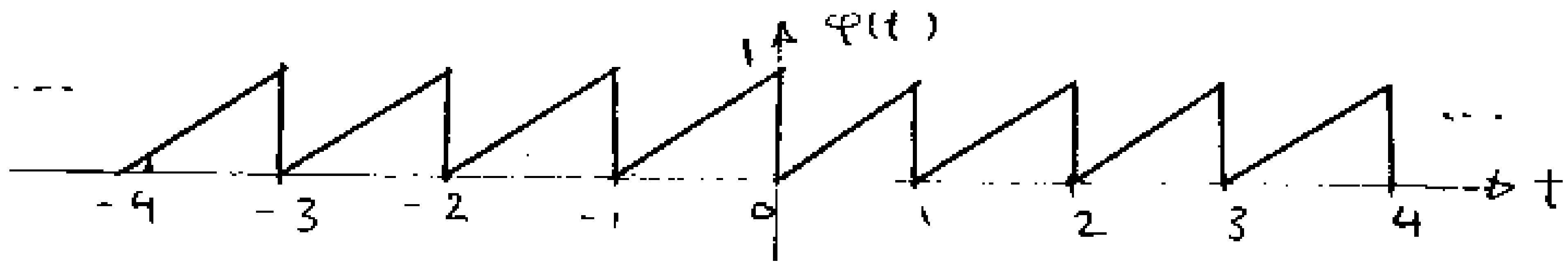


3.4-9)

	a	b	c	d	e	f	g	h	i
Periodic?	yes	yes	no	yes	no	yes	yes	yes	yes
ω_0	1	1	—	π	—	1/70	3/4	1	2
Period	2π	2π	—	2	—	140π	$\frac{8\pi}{3}$	2π	π

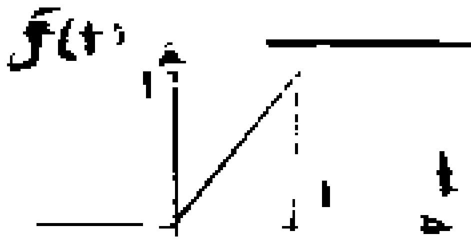
3.4-10)

$$\varphi(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{-1}{n\pi}\right) \sin 2\pi n t$$



$E_e(1) = 1/12$, $E_e(2) = 0.03267$, $E_e(3) = 0.02$, $E_e(4) = 0.014378$

3.4-11)



$$f(t) \simeq \frac{1}{2} x_0(t) - \frac{1}{4} x_1(t) - \frac{1}{8} x_3(t) - \frac{1}{16} x_7(t)$$

$E_e(1) = 0.0833$, $E_e(2) = 0.0204$, $E_e(3) = 6.0052$, $E_e(4) = 0.001302$

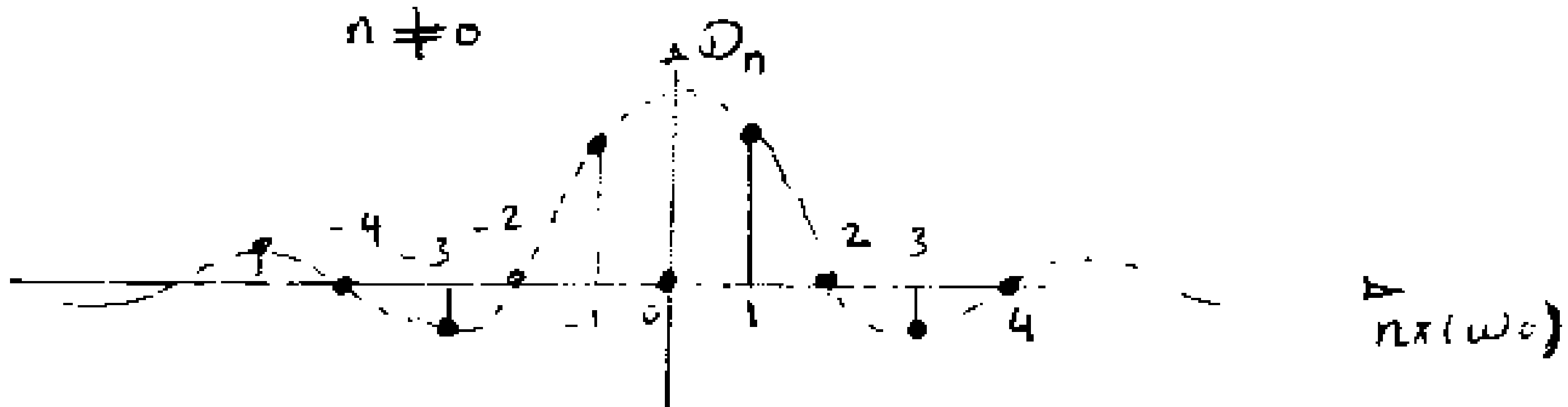
3.4-12)

a) $f(t) = -\frac{3}{2} t + \frac{7}{8}\left(\frac{5}{2} t^3 - \frac{3}{2} t\right) + \cdots$, $E_e(1) = 0.5$, $E_e(2) = 0.28125$

b) $f_b(t) = f\left(\frac{t}{\pi}\right) = -\frac{3}{2}\left(\frac{t}{\pi}\right) + \frac{7}{8}\left[\frac{5}{2}\left(\frac{t}{\pi}\right)^3 - \frac{3}{2}\left(\frac{t}{\pi}\right)\right] + \cdots$

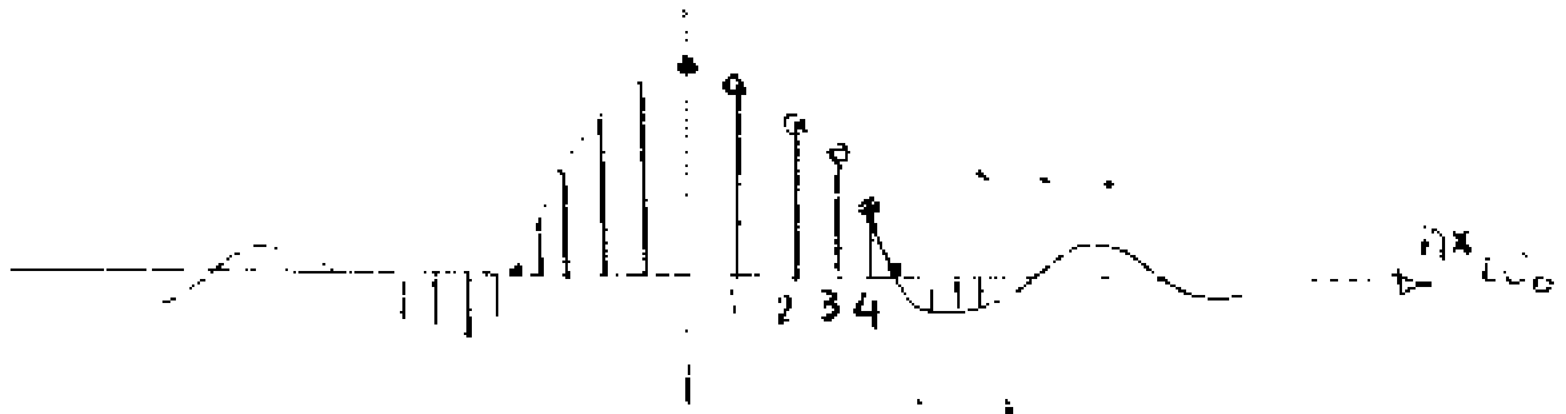
3.5-1)

(a) $f(t) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2}{n\pi} \sin\frac{n\pi}{2}\, e^{j\frac{n\pi}{2}t}$ $\qquad \omega_0 = \frac{\pi}{2}$



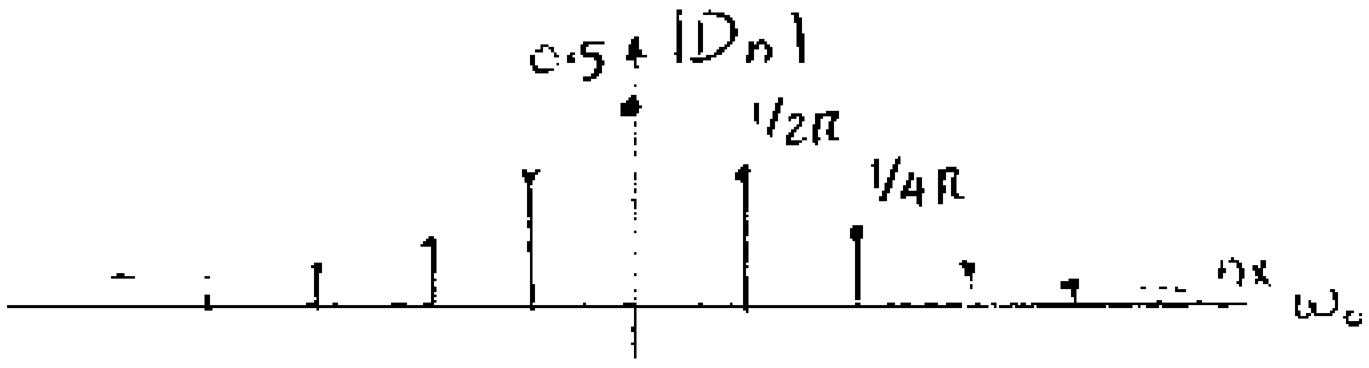
(7)

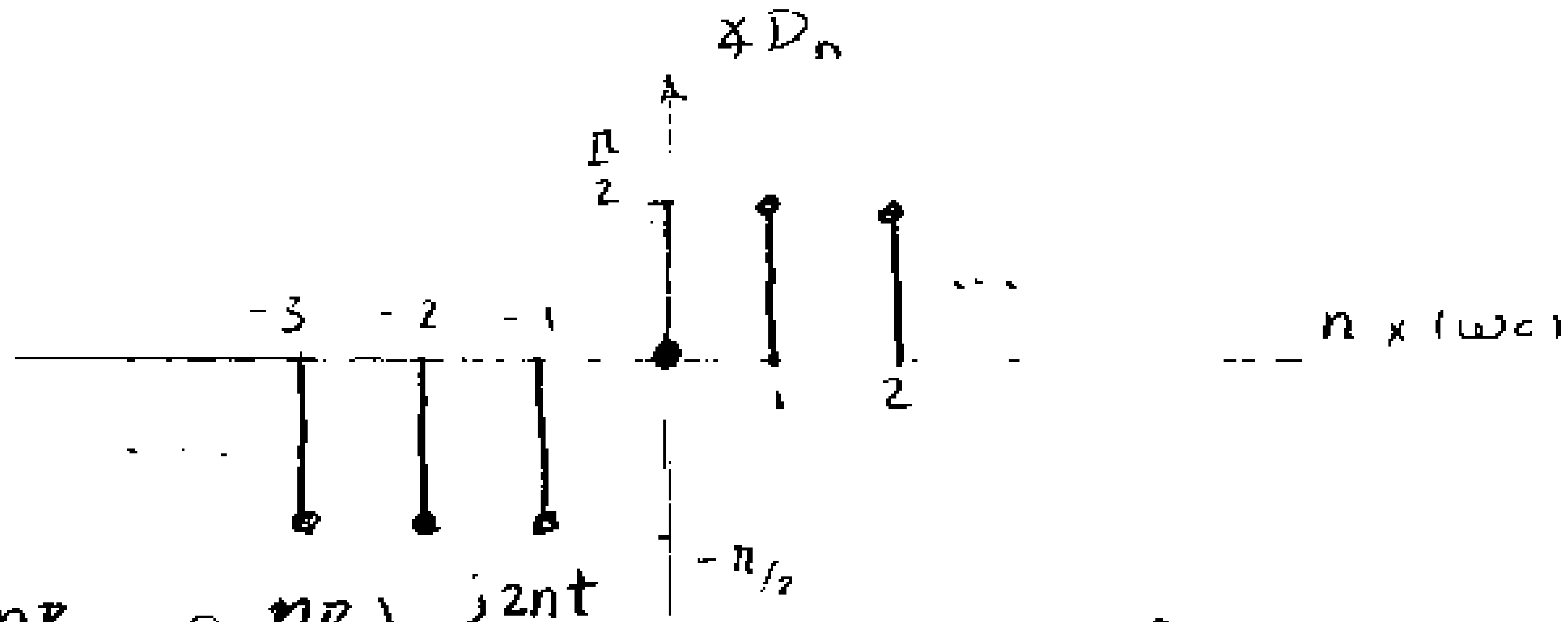
(b) $$b = \sum_{n=-\infty}^{\infty} \frac{1}{n\pi} \sin\left(\frac{n\pi}{5}\right) e^{jnt/5}$$



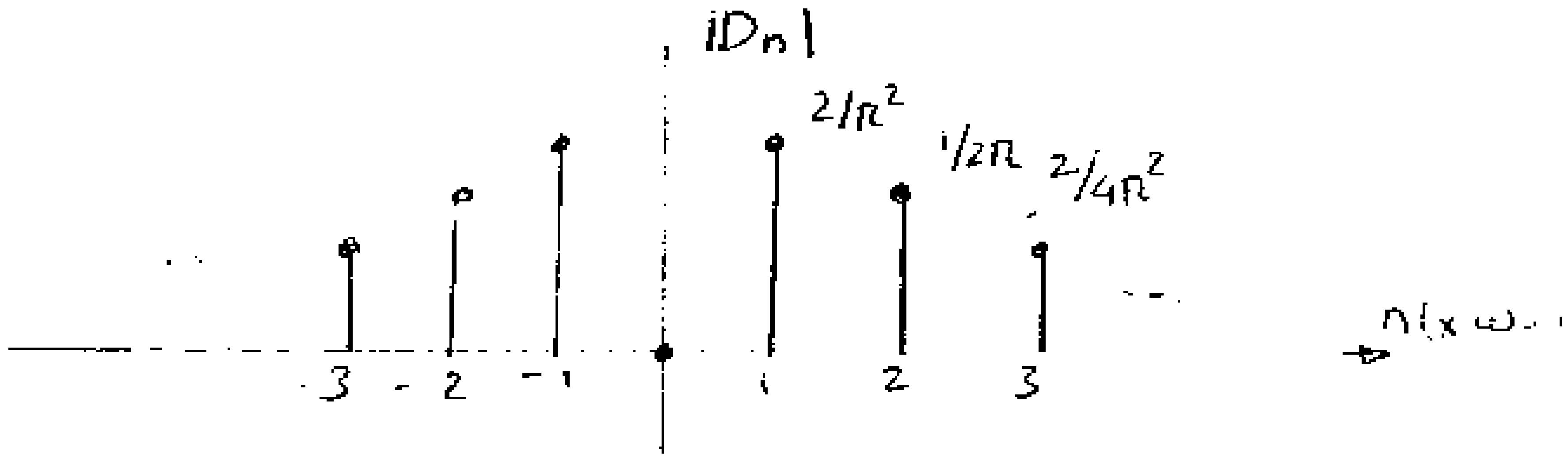
(c) $$f(t) = 0.5 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j}{2\pi n} e^{jnt} \qquad \omega_0 = 1$$

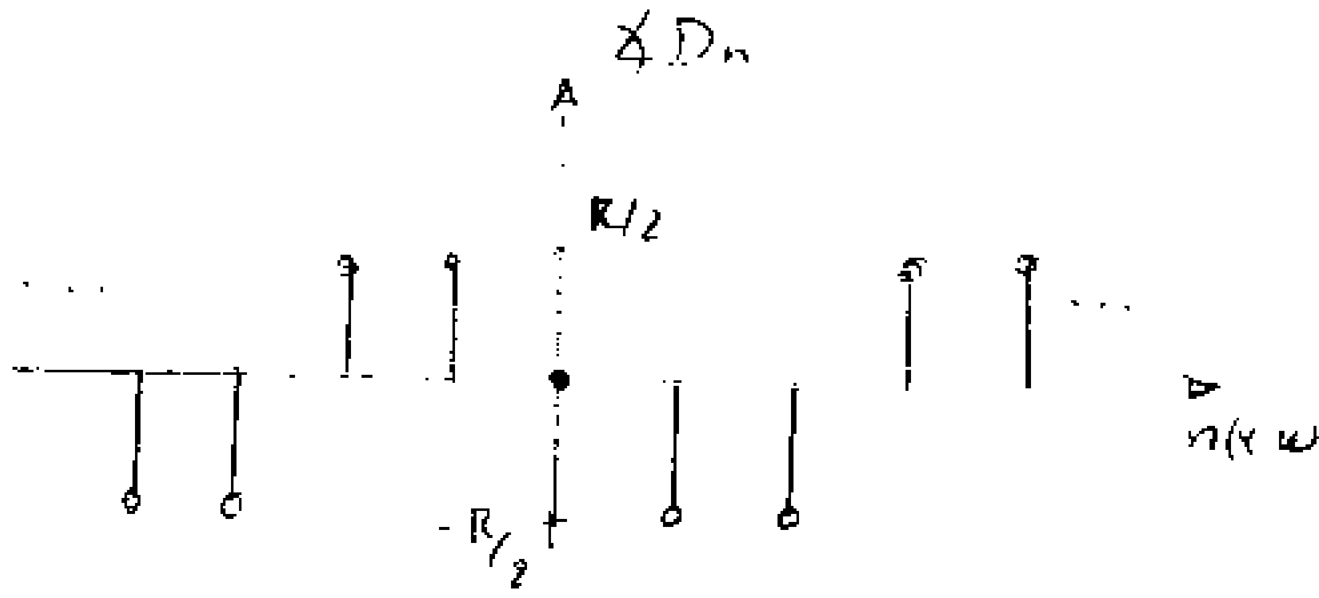
$$\Rightarrow |D_n| = \frac{1}{2\pi n}, \quad n \neq 0, \qquad \measuredangle D_n = \begin{cases} \pi/2 & n>0 \\ -\frac{\pi}{2} & n<0 \end{cases}$$





(d) $$f(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{-j}{n\pi}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right) e^{j2nt} \qquad \omega_0 = 2$$





(e) $$f(t) = \sum_{n=-\infty}^{\infty} \frac{3}{4\pi^2 n^2}\left[e^{-j\frac{2\pi n}{3}}\left(j\frac{2\pi n}{3} + 1\right) - 1\right] e^{j\frac{2\pi n}{3}t}$$

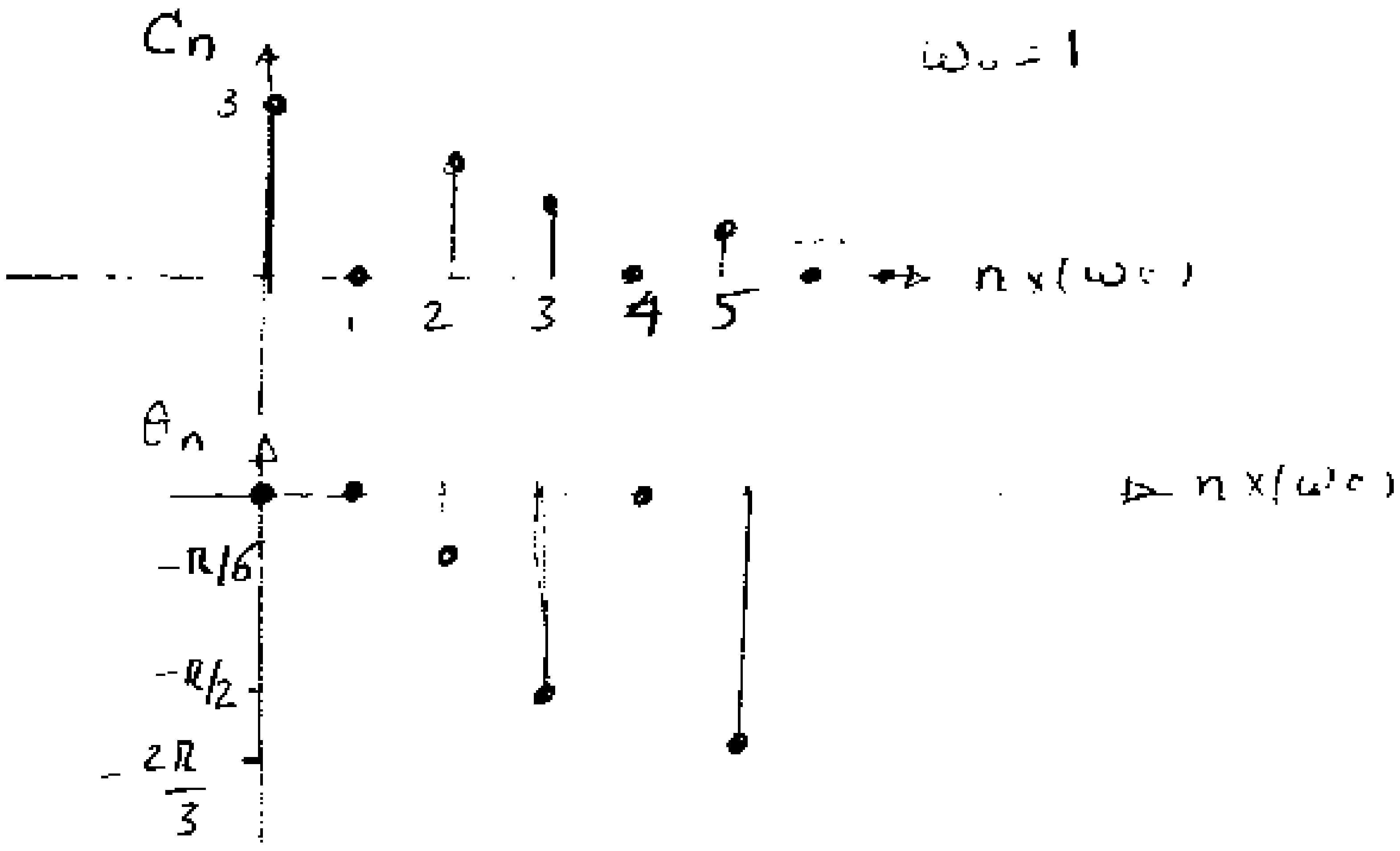
$$|D_n| = \frac{3}{4\pi^2 n^2}\left[\sqrt{2 + \frac{4\pi^2 n^2}{9} - 2\cos\frac{2\pi n}{3} - \frac{4n\pi}{3}\sin\frac{2\pi n}{3}}\right]$$

$$\measuredangle D_n = \tan^{-1}\left(\frac{\frac{2\pi n}{3}\cos\frac{2\pi n}{3} - \sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1}\right)$$

(f) $$f(t) = 0.5 + \sum_{n=-\infty}^{\infty} \frac{3}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - \cos\frac{2\pi n}{3}\right) e^{jn\pi t/3}$$

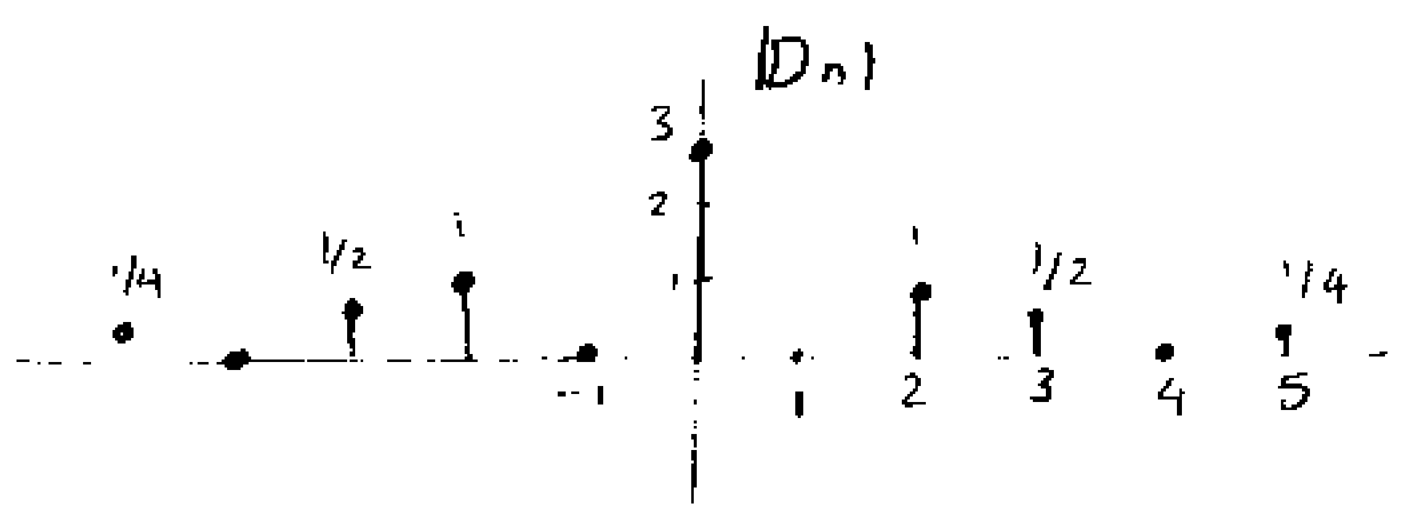
3.5-2)

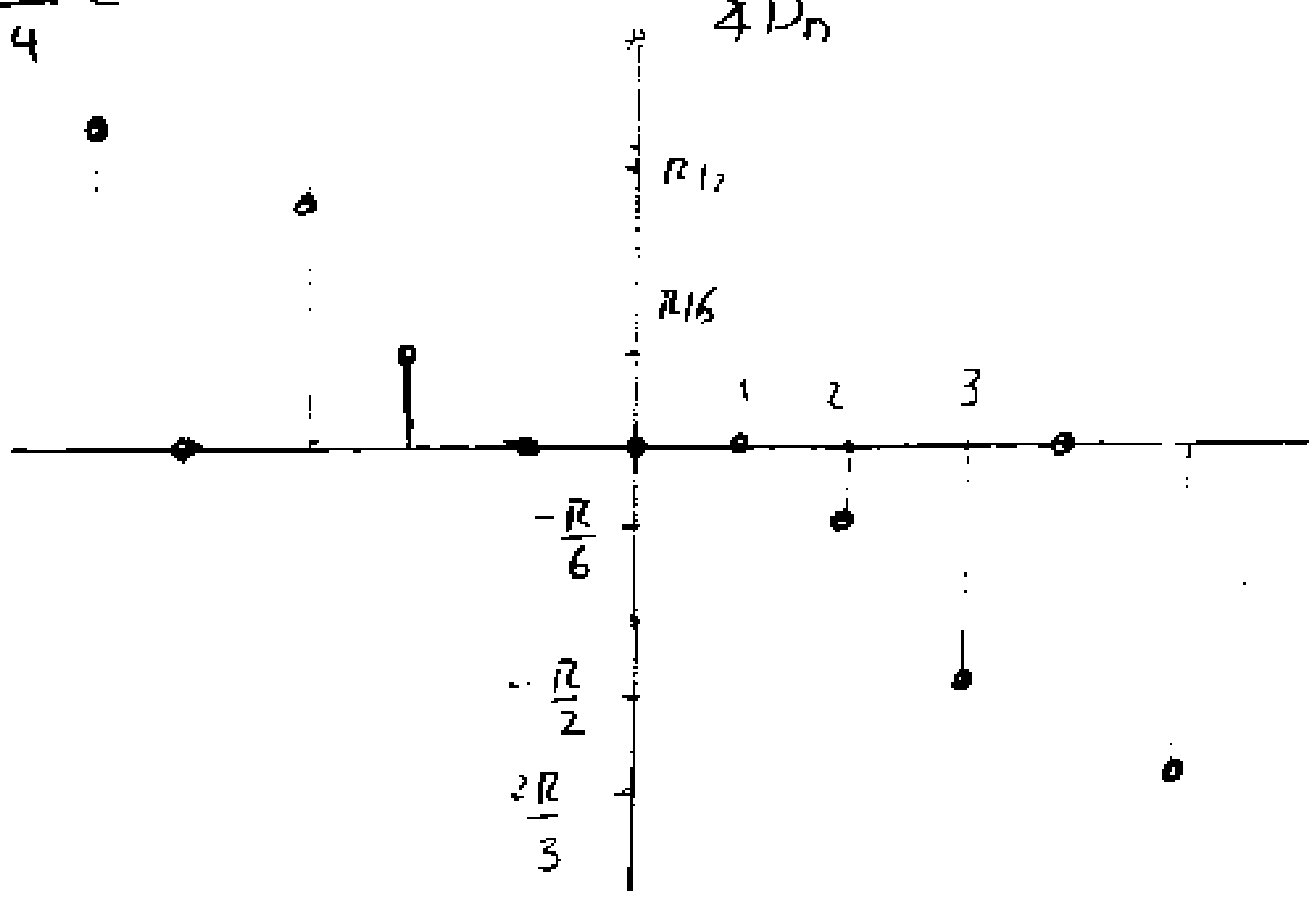
a)



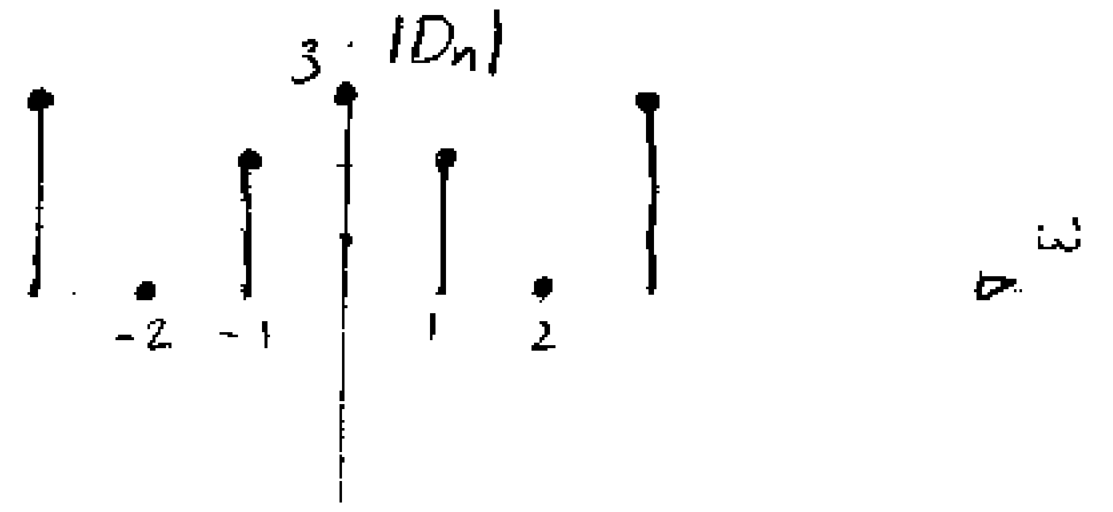
$$f(t) = 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t + \frac{\pi}{3} - \pi\right)$$

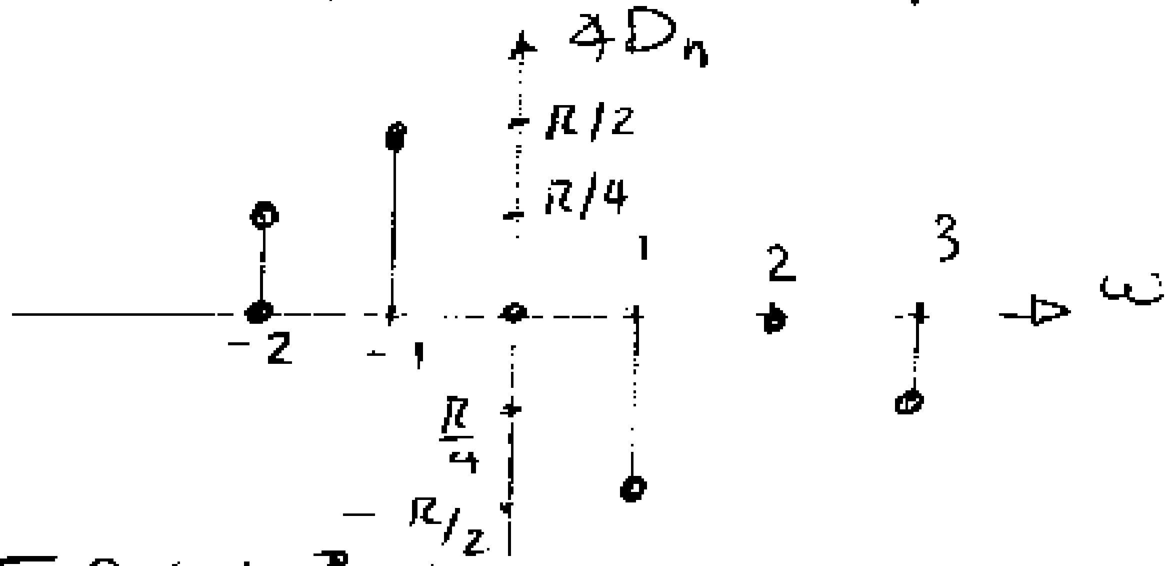
b) $$f(t) = 3 + e^{j(2t-\pi/6)} + e^{-j(2t-\pi/6)} + \frac{1}{2}e^{j(3t-\pi/2)} + \frac{1}{2}e^{-j(3t-\pi/2)} + \frac{1}{4}e^{j(5t-\frac{2\pi}{3})} + \frac{1}{4}e^{-j(5t-2\pi/3)}$$



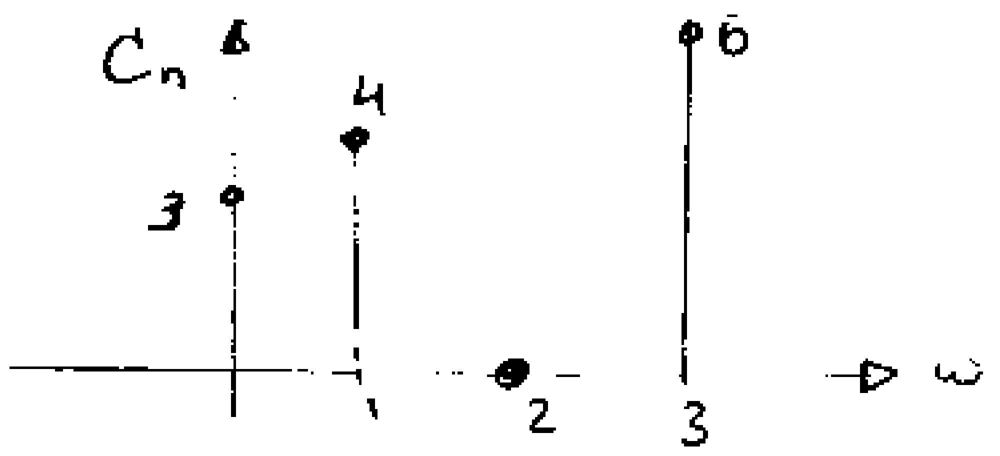


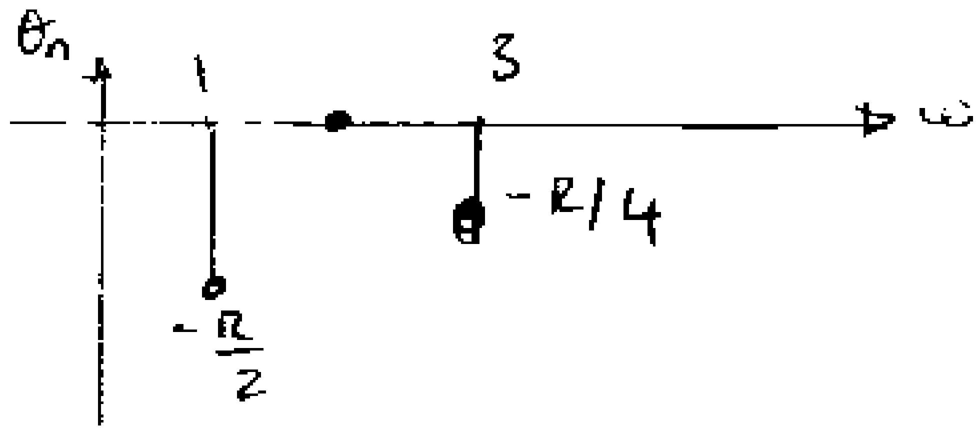
3.5-3) (a) $$f(t) = \left(2\sqrt{2}\,e^{j\pi/4}\right)e^{-j3t} + 2e^{j\pi/2}e^{-jt} + 3 + 2e^{-j\pi/2}e^{jt} + \left(2\sqrt{2}\,e^{-j\pi/4}\right)e^{j3t}$$





(b) $$f(t) = 3 + 4\cos(t - \pi/2) + 4\sqrt{2}\cos(3t - \pi/4)$$





3.5-4) Solved in detailed solution of selected Problems

3.5-5) (a)

From exercise E 3.6a $\quad f(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t \qquad -1 \le t \le 1$

The Power of $f(t)$ is $P_f = \frac{1}{2}\int_{-1}^{1} t^2 dt = 1/5$

Moreover, from Parsval's theorem [Eq. (3.82)]

$$P_f = C_0^2 + \sum_{n=1}^{\infty} \frac{C_n^2}{2} = \left(\frac{1}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{\pi^2 n^2}\right)$$

$$= \frac{1}{9} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{9} + \frac{8}{90} = \frac{1}{5}$$

(b) If the N-term Fourier series is denoted by $x(t)$, then

$$x(t) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi t \quad -1 \le t \le 1$$

The power P_x is required to be 99% $P_f = 0.198$.

Therefore $\quad P_x = \frac{1}{9} + \sum_{n=1}^{N-1} \frac{8}{\pi^2} \cdot \frac{1}{n^4} = 0.198$

For $N=1$, $P_x = 0.1111$, for $N = 2$, $P_x = 0.19323$,
for $N = 3$, $P_x = 0.19837$, which is greater than 0.198,
thus $N = 3$.

3.5-6) (a) From Exercise E 3.6b

$$f(t) = \frac{2A}{\pi}(-1)^{n+1}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi t \qquad -\pi \le t \le \pi$$

The Power of $f(t)$ is $\quad P_f = \frac{1}{2}\int_{-1}^{1}(At)^2 dt = A^2/3$

Moreover, from Parsval's theorem

$$P_f = C_0^2 + \sum_{n=1}^{\infty}\frac{C_n^2}{2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{4A^2}{\pi^2 n^2} = \frac{2A^2}{\pi^2}\sum_{n=1}^{\infty} = \frac{A^2}{3}$$

(b) If the N-term Fourier series is denoted by $x(t)$, then

$$x(t) = \frac{2A}{\pi}(-1)^{n+1}\sum_{n=1}^{N}\frac{1}{n}\sin n\pi t \qquad -\pi \le t \le t$$

The Power P_x is required to be $0.9\frac{A^2}{3} = 0.3A^2$.

Therefore $P_x = \frac{1}{2}\sum_{n=1}^{N}\frac{4A^2}{\pi^2 n^2} = 0.3A^2$

by inspection

$\Rightarrow$ $N = 6$

3.5-7) The Power of rectified Sine wave is the same as that of a sine wave, that is, $1/2$. Thus $P_f = 0.5$. Let the $2N+1$ term truncated Fourier series be denoted by $\hat{f}(t)$. The Power $P_{\hat{f}}$ is required to be $0.9975\,P_f = 0.49875$. Using the Fourier series Coefficients in excersise E3.10, we have

$$P_{\hat{f}} = \sum_{n=-N}^{N}|D_n|^2 = \frac{4}{\pi^2}\sum_{n=-N}^{N}\frac{1}{(1-4n^2)^2} = 0.49875$$

Direct calculations using the above equation gives $P_{\hat{f}} = 4/\pi^2 = 0.4053$ for $N=0$ (only dc), $P_{\hat{f}} = 0.49535$ for $N=1$ (3 terms), and $P_{\hat{f}} = 0.49895$ for $N=2$ (5 terms)

Thus, a 5-term Fourier series yields a signal whose Power is 99.79% of the Power of the rectified sine wave. The Power of the error in the approximation of $f(t)$ is only 21% of the signal Power P_f.

3.6-1) $$y(t) = \sum_{n=-\infty}^{\infty}\frac{j\,1.08n}{(1+j4n)(-\omega^2+3+j2\omega)}e^{j2nt}$$