chapter 5 (Solution to selected Problem)

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5.1-1

The bandwidth of $f_1(t)$ and $f_2(t)$ are $B_1 = 100$ KHz and $B_2 = 150$ KHz respectively. The Nyquist rate is twice the bandwidth. So the Nyquist rate for $f_1(t)$ is 200 KHz and for $f_2(t)$ is 300 KHz.
For $f_1^2(t)$, the band width can be calculated using frequency convolution property. $f_1^2(t) \Longleftrightarrow \frac{1}{2\pi} F_1(\omega) * F_1(\omega)$, and from the width property of convolution, the bandwidth of $f_1^2(t)$ is twice the bandwidth of $f_1(t)$ and that of $f_2^3(t)$ is three times the bandwidth of $f_2(t)$. Similarly the bandwidth of $f_1(t) \cdot f_2(t)$ is $B_1 + B_2$. Therefore the Nyquist rate for $f_1^2(t)$ is 400 KHz, for $f_2^3(t)$ is 900 KHz, and for $f_1(t) \cdot f_2(t)$ is 500 KHz.

5.1-2 To find the Nyquist sampling rate and period the first step is to find the signal frequency bandwidth. So we have to find the Fourier transform of the different signals:

(a) $\text{sinc}^2(100\pi t) \xleftrightarrow{\mathcal{F}} 0.01\, \Delta\left(\frac{\omega}{400\pi}\right)$ using Fourier table

The bandwidth of this signal is 200π rad/s or 100 Hz. Then the Nyquist rate is 200 Hz and Nyquist period is $T_s = \frac{1}{200}$ sec $= 0.005$ sec.

(b) $0.01\, \text{sinc}^2(100\pi t)$ has exactly the same bandwidth as the signal in part (a) since the Fourier transform is scaled by 0.01 and the bandwidth is still 100 Hz. Then Nyquist rate is 200 Hz and Nyquist period is 0.005 sec.

(c) $\text{sinc}(100\pi t) + 3\,\text{sinc}^2(60\pi t) \overset{\mathcal{F}}{\Longleftrightarrow} 0.01\, \text{rect}\left(\frac{\omega}{200\pi}\right) + \frac{1}{20}\, \Delta\left(\frac{\omega}{240\pi}\right)$

The bandwidth of $\text{rect}\left(\frac{\omega}{200\pi}\right)$ is 50 Hz and that of $\Delta\left(\frac{\omega}{240\pi}\right)$ is 60 Hz. The bandwidth of the sum is the higher of the two, that is 60 Hz. The Nyquist rate is 2×60 Hz $= 120$ Hz and Nyquist period is $\frac{1}{120}$ sec.

(d) $\text{sinc}(50\pi t) \xleftrightarrow{\mathcal{F}} 0.02\, \text{rect}\left(\frac{\omega}{100\pi}\right)$
$\text{sinc}(100\pi t) \xleftrightarrow{\mathcal{F}} 0.01\, \text{rect}(\omega/200\pi)$

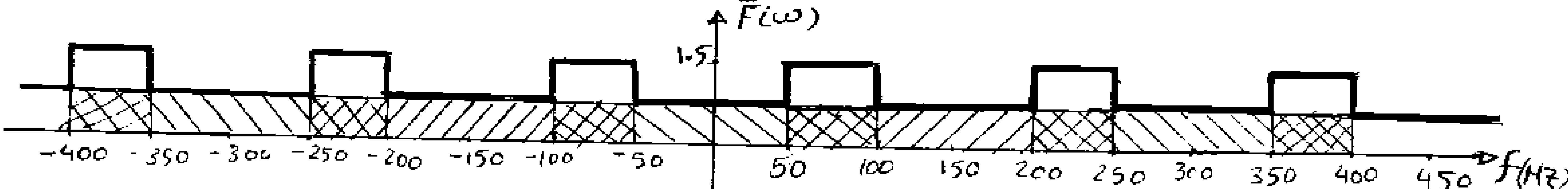
The two signals have bandwidths 25 Hz and 50 Hz respectively.

The bandwidth of product of two signals is the addition of two bandwidths. So $B = 25 + 50 = 75$ Hz. Therefore the Nyquist rate is 150 Hz and the Nyquist period is $1/150$ sec.

5.1-3 $\quad f(t) = \text{sinc}(200\pi t) \xleftrightarrow{\mathcal{F}} 0.005\, \text{rect}\left(\frac{\omega}{400\pi}\right)$

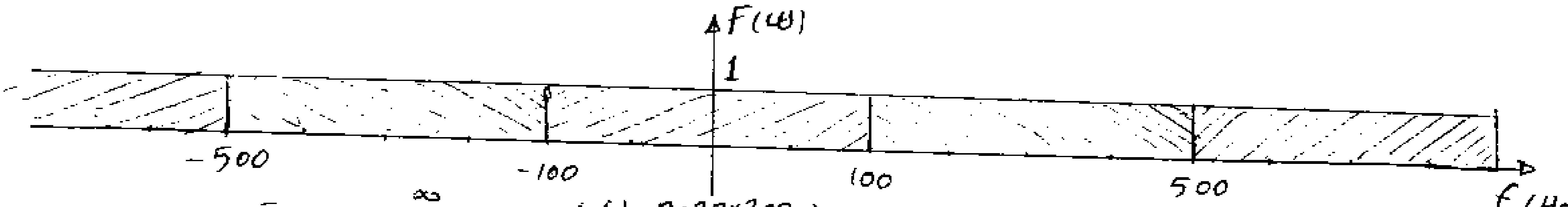
The bandwidth of the signal is 100 Hz (200π rad/sec), the Nyquist rate is 200 Hz. So any sampling frequency less than 200 Hz results aliasing (under sampling) and any sampling frequency more than 200 Hz results oversampling.

For $f_s = 150$ Hz we have the following frequency spectrum if we sample the signal using uniformly spaced impulse train:



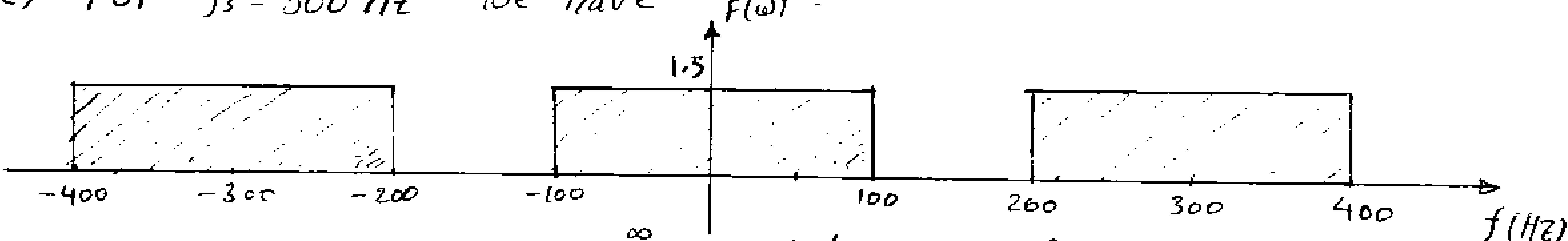
$$\bar{F}(\omega) = f_s \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s) = 150 \sum_{n=-\infty}^{\infty} 0.005\, \text{rect}\left(\frac{\omega - n\omega_s}{400\pi}\right) = 0.75 \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{\omega - n\omega_s}{400\pi}\right)$$

(b) For $f_s = 200$ Hz we have:



$$\bar{F}(\omega) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{\omega - n \times 2\pi \times 200}{400\pi}\right)$$

(c) For $f_s = 300$ Hz we have



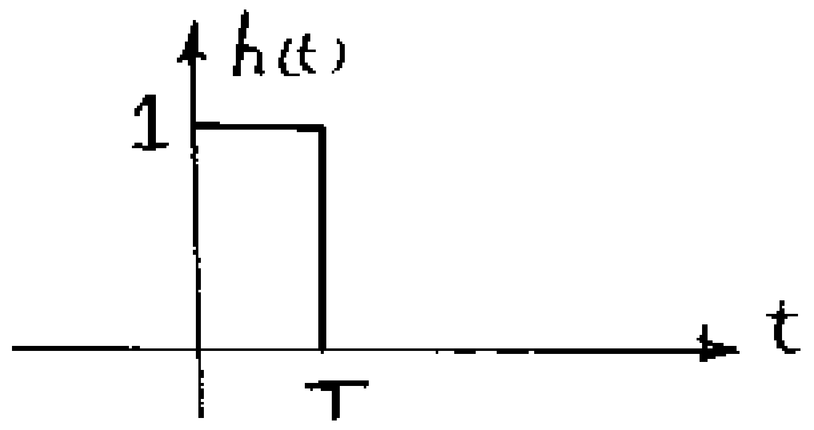
$$\bar{F}(\omega) = 1.5 \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{\omega - n \times 2\pi \times 300}{400\pi}\right)$$

It can be seen that undersampling changes the original signal spectrum if we pass the sampled signal through a lowpass signal.

5.1-4 (a) The impuls response of the systen can be found by the block diagram:

when we apply impulse $\delta(t)$ to the input, the output of the summer is $\delta(t)-\delta(t-T)$. This is the input to the integrator. So, $h(t)$ is the output of the integrator

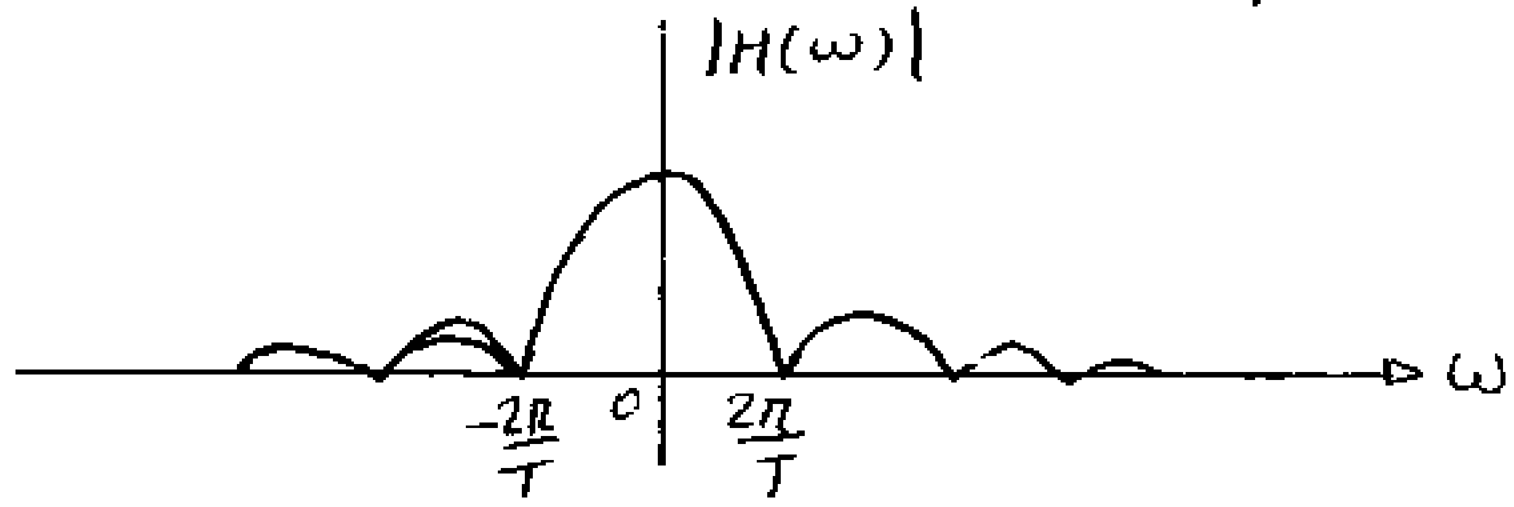
$$h(t)=\int_0^t [\delta(\tau)-\delta(\tau-T)]\,d\tau = u(t)-u(t-T) = \text{rect}\left(\frac{t-T/2}{T}\right)$$



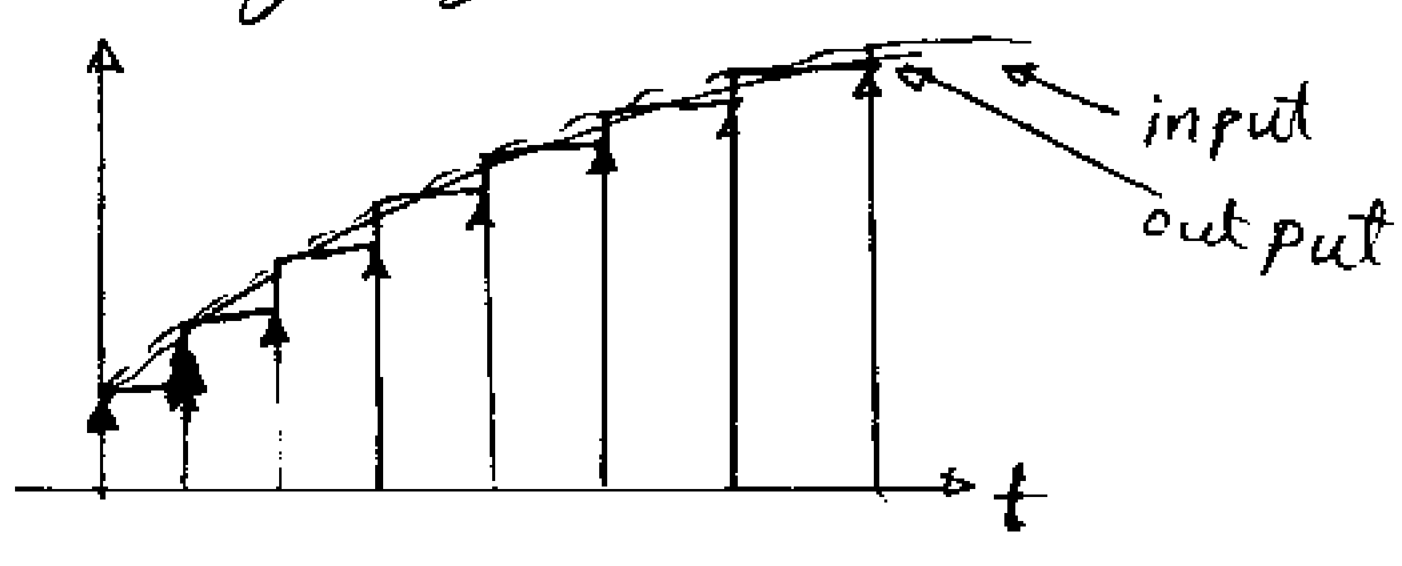
(b) Using Fourier properties and Fourier transform of rectangular pulse $\text{rect}\left(\frac{t}{T}\right)$ we have:

$$H(\omega) = T\,\text{sinc}\left(\frac{\omega T}{2}\right)e^{-j\omega T/2}$$

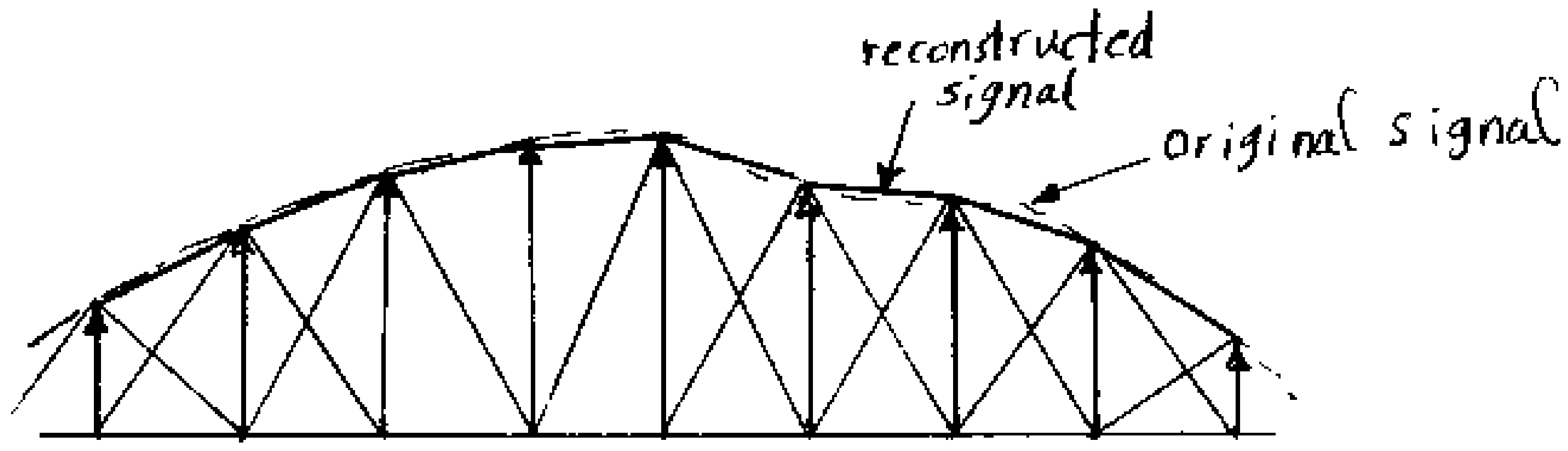
and $$|H(\omega)| = T\left|\text{sinc}\left(\frac{\omega T}{2}\right)\right|$$



This filter is a non ideal lowpass filter with bandwidth of $\frac{2\pi}{T}$. The impulse response of the circuit is a rectangular pulse. When a sampled signal is applied at the input, each sample generates a rectangular pulse at the output, proportional to the corresponding sample value. Hence the output is a staircase approximation of the input as shown in the following figure



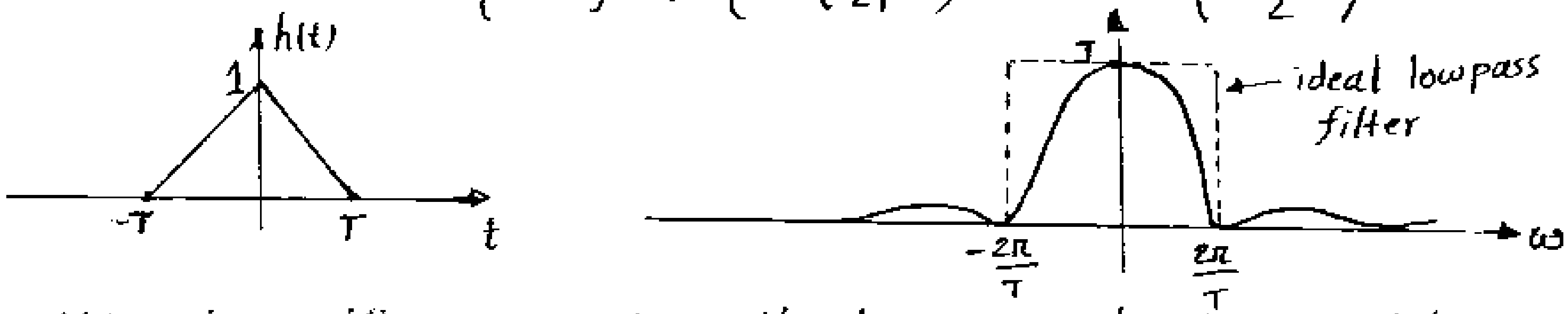
5.1-5 (a) The following figure shows a signal reconstructed using first-order hold circuit



Each sample generates a triangle of width 2T and centered at the sampling instant. The height of the triangle is equal to the sample value. The resulting signal consists of straight line segments joining the sample tops.

(b) The following figures show the impulse response and the transfer function (Fourier transform of the impulse response). we have:

$$H(\omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\left\{\Delta\left(\frac{t}{2T}\right)\right\} = T\,\mathrm{sinc}^2\left(\frac{\omega T}{2}\right)$$



Because $H(\omega)$ is positive for all ω, it also represents the amplitude response.

(c) A minimum of T seconds delay is required to make $h(t)$ causal. Such a delay would cause the reconstructed signal to be delayed T seconds.

(d) When the input to the first filter is $\delta(t)$, then as shown in problem 5.1-4 its output is a rectangular pulse $p(t) = u(t) - u(t-T)$ shown in solution to the previous problem. This pulse $p(t)$ is applied to the input of the second identical filter. The output of the second filter is $p(t) - p(t-T) = u(t) - 2u(t-T) + u(t-2T)$, which is applied to the integrator. The output $h(t)$ of the integrator is the area under $p(t) - p(t-T)$, which is:

$$h(t) = \int_0^t [u(\tau) - 2u(\tau - T) + u(\tau - 2T)]\,d\tau = \Delta\left(\frac{t-T}{T}\right)$$

5.1-6 The signal $f(t) = \text{sinc}(200\pi t)$ is sampled by a rectangular pulse sequence $P_T(t)$ whose period is 4 msec so that the fundamental frequency (which is also the sampling frequency) is 250 Hz. Hence, $\omega_s = 500\pi$. The Fourier series for $P_T(t)$ is given by

$$P_T(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\omega_s t$$

where $C_0 = 0.2,\ C_1 = 0.374,\ C_2 = 0.303,\ C_3 = 0.202,\ C_4 = 0.093, \dots$

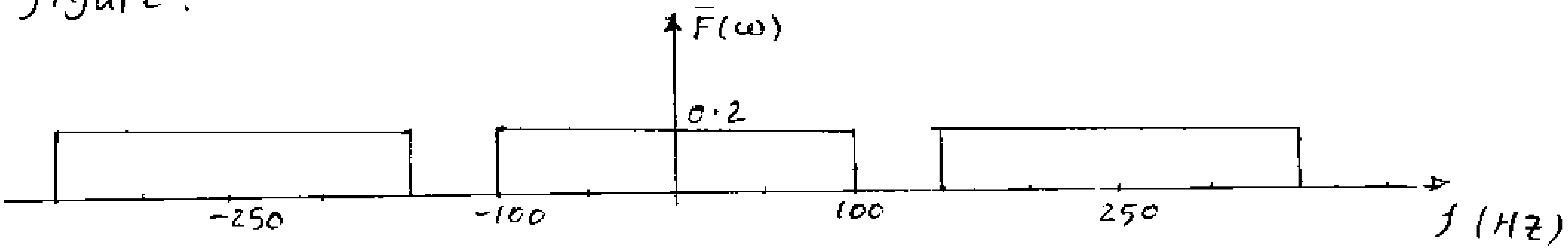
Consequently

$$\bar{f}(t) = f(t)P_T(t) = 0.2 f(t) + 0.374 f(t)\cos 500\pi t + 0.303 f(t)\cos 1000\pi t + 0.202 f(t)\cos 1500\pi t + \dots$$

and

$$\bar{F}(\omega) = 0.2\, F(\omega) + 0.187\left[F(\omega - 500\pi) + F(\omega + 500\pi)\right] + 0.151\left[F(\omega - 1000\pi) + F(\omega + 1000\pi)\right] + 0.101\left[F(\omega - 1500\pi) + F(\omega + 1500\pi)\right] + \dots$$

In the case $F(\omega) = 0.005\,\text{rect}\left(\frac{\omega}{400\pi}\right)$, $\bar{F}(\omega)$ is shown in the following figure:



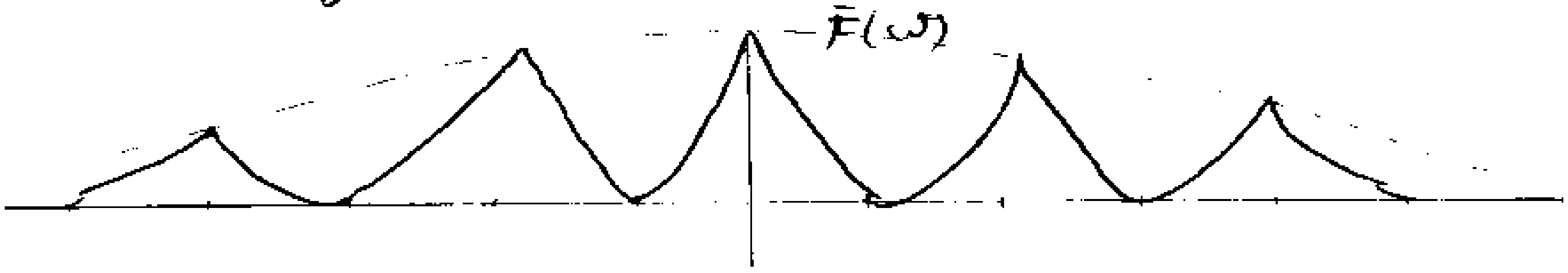
There is no overlap between cycles, and $F(\omega)$ can be recovered using an ideal lowpass filter of bandwidth 100 Hz.

Because the spectrum $\bar{F}(\omega)$ has a zero value in the band from 100 to 150 Hz, we can use an ideal lowpass filter of bandwidth B (Hz) where $100 < B < 150$. But if $B > 150$, the filter will pick up the unwanted spectral components from the next cycle, and the output will be distorted.

5.1-7 The signal $f(t)$, when sampled by an impulse train, results in the sampled signal $f(t)\delta_T(t)$ (as shown in Fig 5.1d). If this signal is transmitted through a filter whose impulse response is $h(t) = p(t) = \text{rect}(t/0.025)$, then each impulse in the input will generate a pulse $p(t)$, resulting in the desired sampled signal shown in figure P5.1-7. Moreover, the spectrum of the impulse train $f(t)\delta_T(t)$ is $\frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)$. Hence, the output of the filter is

$$\bar{F}(\omega) = H(\omega)\cdot\left[\frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)\right]$$

where $H(\omega) = P(\omega) = 0.025\,\text{sin}\left(\frac{\omega}{80}\right)$, the Fourier transform of $\text{rect}\left(\frac{t}{0.025}\right)$. The following figure shows this spectrum consisting of the repeating spectrum $F(\omega)$ multiplied by $H(\omega) = 0.025\,\text{sinc}\left(\frac{\omega}{80}\right)$. Thus, each cycle is somewhat distorted.



To recover the signal $f(t)$ from the flat top samples, we reverse the process of $h(t)$. First, we pass the sampled signal through a filter with transfer function $1/H(\omega)$. This will yield the signal sampled by impulse train. Now we pass this signal through an ideal lowpass filter of bandwidth B Hz to obtain $f(t)$.

5.1-8 (a) The bandwidth is 15 KHz. The Nyquist rate is 30 KHz.

(b) $65536 = 2^{16}$, so that 16 binary digits are needed to encode each sample.

(c) $30000 \times 16 = 480000$ bits/s

(d) $44100 \times 16 = 705600$ bits/s

5.1-9

(a) The Nyquist rate is $2 \times 4.5 \times 10^6 = 9$ MHz. The actual rate is $1.2 \times 9 = 10.8$ MHz.

(b) $1024 = 2^{10}$, so that 10 bits or binary pulses are needed to encode each sample.

(c) $10.8 \times 10^6 \times 10 = 108 \times 10^6$ or 108 Mbits/s.

5.1-10 Assume a signal $f(t)$ that is simultaneously time-limited. Let $F(\omega) = 0$ for $|\omega| > 2\pi B$. Therefore $F(\omega) \cdot \text{rect}(\omega/4\pi B') = F(\omega)$ for $B' > B$. Therefore from the time-Convolution property (4.42)

$$f(t) = f(t) * [\, 2B' \,\text{sinc}\,(2\pi B' t)]$$
$$= 2B' f(t) * \text{Sinc}\,(2\pi B' t)$$

Because $f(t)$ is time limited, $f(t) = 0$ for $|t| > T$. But $f(t)$ is equal to convolution of $f(t)$ with $\text{sinc}(2\pi B' t)$ which is not time limited. It is impossible to obtain a time-limited signal with non-timelimited signal.

5.2-1 $T_0 = \frac{1}{F_0} = \frac{1}{50} = 20$ msec

$B = 10000$ Hence $F_s \geq 2B = 20000$

$T = 1/F_s = \frac{1}{20000} = 50\,\mu\text{sec}.$

$$N_0 = \frac{T_0}{T} = \frac{20 \times 10^{-3}}{50 \times 10^{-6}} = 400$$

Since N_0 must be a power of 2, we choose $N_0 = 512$.
Also $T = 50\,\mu\text{sec}$, and $T_0 = N_0 T = 512 \times 50\,\mu\text{sec} = 25.6$ msec.
$F_0 = 1/T_0 = 39.0625$ Hz. Since $f(t)$ is of 10 msec duration, we need zero padding over 15.6 msec. Alternatively, we could also have used $T = \frac{20 \times 10^{-3}}{512} = 39.0625\,\mu\text{sec}$

This gives $T_0 = 20$ msec, $F_0 = 50$ Hz, And
$$F_s = \frac{1}{T} = 25600 \text{ Hz}$$

There are also other possibilities of reducing T as well as increasing the frequency resolution

5.2-3

$f(t) = e^{-t}u(t) \qquad F(\omega) = \frac{1}{j\omega+1}$

(a) We take the folding frequency F_s to be the frequency where $|F(\omega)|$ is 1% of its peak value, which happens to be 1 (at $\omega=0$). Hence,

$$|F(\omega)| \approx \frac{1}{\omega} = 0.01 \Rightarrow \omega = 2\pi B = 100$$

This yields $B = 50/\pi$, and $T \le 1/2B = \pi/100$. Let us round T to 0.03125, resulting in 32 sampling per second. The time constant of e^{-t} is 1. For T_0, a reasonable choice is 5 to 6 time constants or more. Value of $T_0 = 5$ or 6 results in $N_0 = 160$ or 192, neither of which is a power of 2. Hence, we choose $T_0 = 8$, resulting in $N_0 = 32 \times 8 = 256$, which is a power of 2.

(b)

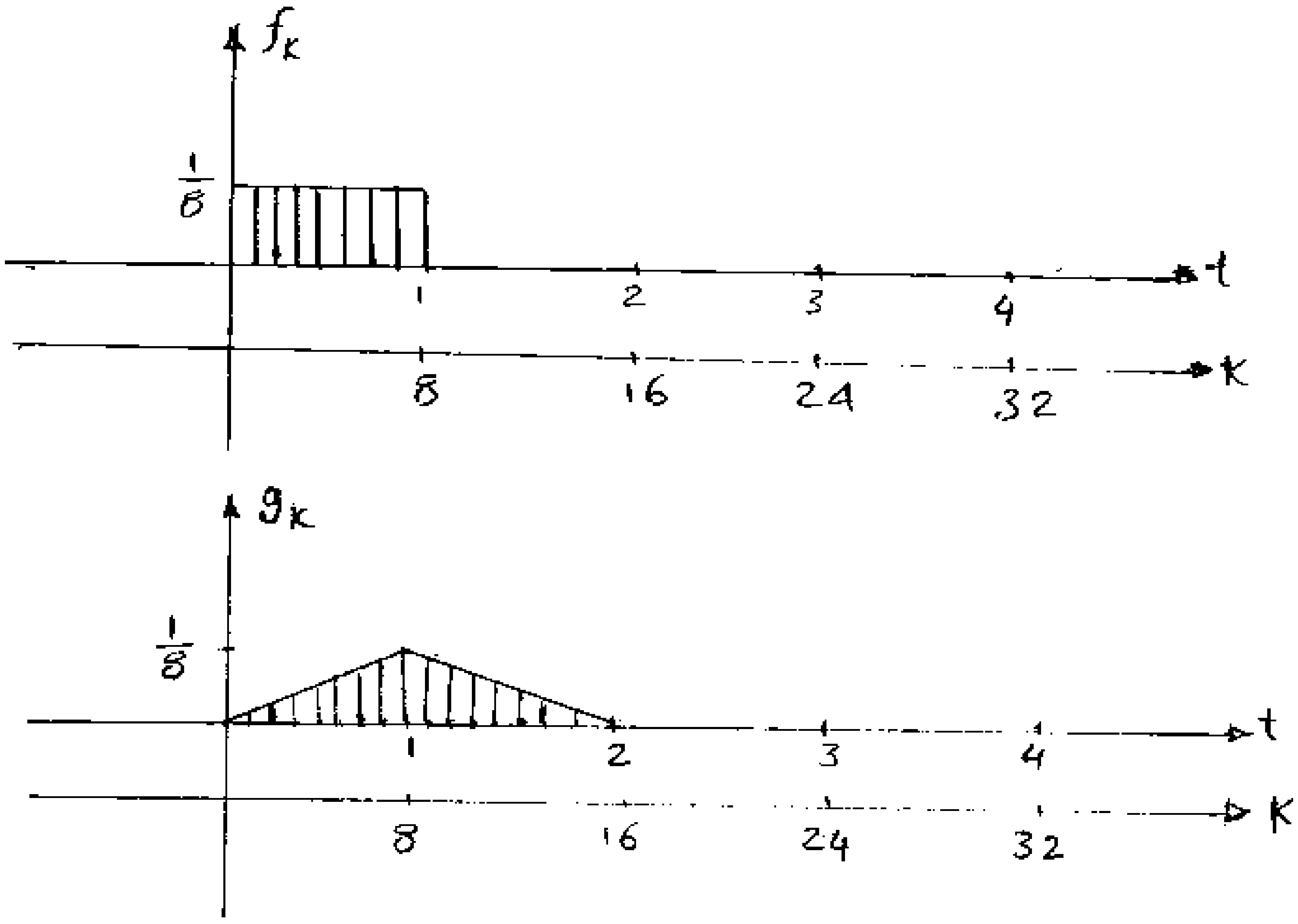
$$|F(\omega)| = \frac{1}{\sqrt{\omega^2+1}} \simeq \frac{1}{\omega} \qquad \omega \gg 1$$

we take the folding frequency F_s to be 99% energy frequency as explained in Example 4.16. From the results in Example 4.16, (with $a=1$) we have:

$$\frac{0.99\pi}{2} = \tan^{-1}\frac{W}{a} \Rightarrow W = 63.66a = 63.66 \text{ rad/sec.}$$

This yields $B = \frac{W}{2\pi} = 10.13$ Hz. Also $T \le 1/2B = 0.04936$. This results in the sampling rate $\frac{1}{T} = 20.26$ Hz. Also $T_0 = 8$ as explained in part (a). This yields $N_0 = 20.26 \times 8 = 162.08$, which is not a power of 2. Hence, we choose the next higher value, that is $N_0 = 256$, which yields $T = 0.03125$ and $T_0 = 8$, the same as in part (a).

5.2-5 The width of $f(t)$ and $g(t)$ are 1 and 2 respectively. Hence the width of the convolution signal is $1+2=3$. This means we need to zeropad $f(t)$ for 2 seconds, and $g(t)$ for 1 second, making $T_0 = 3$ for both signals. Since $T = 0.125$ $N_0 = 3/(0.125) = 24$. N_0 must be a power of 2. Choose $N_0 = 32$. This permits us to adjust T_0 ~~to~~ to 4. Hence $T = 0.125$ and $T_0 = 4$.



f_k
$\frac{1}{8}$
1
2
3
4
t
8
16
24
32
K
g_k
$\frac{1}{8}$
1
2
3
4
t
8
16
24
32
K