EE 3TP4: Signals and Systems
Lab 5: Control of a Servomechanism

Objective

To identify the plant model of a servomechanism, and explore the trade-off between rise time and overshoot incurred in proportional control of the servo(mechanism).

Assessment

Your mark for this lab will be assessed on your ability to complete each section. The marks for each section are indicated at the beginning of the section. Please demonstrate your results to the TAs and have your mark recorded.

Please attend the lab section to which you have been assigned.

1 Description of Laboratory Equipment

In this lab we will deal with an angular positioning system based around a DC motor. Such schemes are often used to position heavy or difficult to move objects using a 'command tool' which is easy to move. For example, moving the control surfaces of an aircraft using the lever in the cockpit.

In our system, the plant is the motor (and its associated electronics). The input to the plant is a control signal $c(t)$ and the output of the plant is a voltage which is proportional to the angle of the motor shaft, $\theta(t)$. Using information about the structure of the motor and Newtonian mechanics, the operation of the motor can be described by the following differential equation:

$$J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} = K_m c(t), \tag{1}$$

where J is the rotational inertia of the motor, B is the damping in the motor structure, and K_m is the (internal) gain of the motor. Taking Laplace transforms of both sides of (1) we obtain the transfer function of the plant:

$$s^2 J\Theta(s) + sB\Theta(s) = K_m C(s) \tag{2}$$

$$\implies G(s) = \frac{\Theta(s)}{C(s)} = \frac{A}{s(s\tau_m + 1)}, \tag{3}$$

where $A = K_m/B$ and $\tau_m = J/B$. In the first part of this lab we will identify A and τ_m, as these are not known in advance in typical industrial applications. Following this system identification phase, we will explore the a design trade-off in proportional control of this system.

2 Familiarize yourself with the Equipment

2.1 Equipment list

The laboratory equipment you will require for this laboratory consists of:

i) Signal generator

ii) Op-amp module: This module will be used to construct the controller for the servomechanism.

iii) Servomechanism electronics unit (SEU): This unit contains the circuits which drive the motor and those which process the sensor output. As far as the control design is concerned, the 'plant' is the cascade of the SEU and the motor. **Please note that you must always set the motor switch to 'inhibit' before turning the SEU on or off.**

iv) DC motor: Note that the flange indicates the angle of the shaft.

v) Scope

vi) Digital multi-meter

vii) Components for the op-amp module, including

- Five 10 kΩ resistors (orange)
- Two 100 kΩ resistors (yellow)
- One 100 kΩ potentiometer.
- Four flat conductors
- Several cables with banana plug connectors
- Two ring/banana converters

Please identify each component and notify your TA of any missing equipment.

2.2 Familiarization exercises

To familiarize yourself with the equipment, perform the following simple tests.

i) Set the motor switch on the SEU to inhibit, and set the resolution to one revolution.

ii) Switch on the SEU.

iii) Rotate the disk on top of the motor one revolution (with your finger) until the red 'uncal' light goes off. You must do this every time you switch on the SEU.

iv) Continue to rotate the disk and observe that the LEDs count upwards from 0° to 360°. Observe that the range of the counter is 0 to 2399.
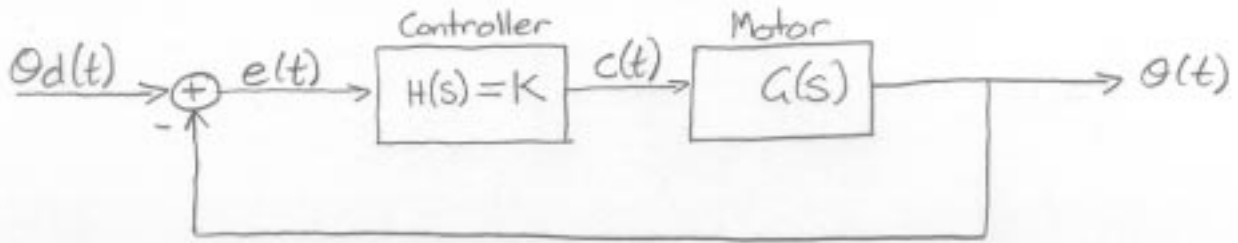
Figure 1: Unity feedback system with proportional control.

v) Connect the ground of the SEU to that of the multi-meter and connect the angle output to the voltage input of the multi-meter. Set the scale to 20V.

vi) Rotate the disk again and observe that the output voltage goes from -10V to 10V.

vii) Disconnect the angle output from the multi-meter and connect the rate output to the multi-meter.

viii) Rotate the disk with your finger in different directions and different speeds and observe the voltage output.

ix) Leave the motor switch at inhibit, and connect the +12V supply on the SEU to the command input. Set the motor switch to enable and observe the direction and speed of rotation and the voltage of the rate output.

x) Set the motor switch to inhibit. Connect the −12V supply on the SEU to the command input. Set the motor switch to enable and observe.

xi) Set the motor switch to inhibit, and set the resolution to two revolutions. Disconnect the multi-meter.

3   Closed Loop System Identification

The plant $G(s)$ in (3) is not bounded-input bounded-output stable, because it has a pole at $s = 0$. Although the system is 'marginally stable' in a certain sense, it is very difficult to identify A and τ_m unless the system is stabilized. In this section we will show how to set up a stable closed loop system, and then identify A and τ_m.

We will construct a stable closed loop by using a simple proportional controller, as shown in Figure 1.[1] Since $G(s) = \frac{A}{s(s\tau_m+1)}$ and $H(s) = K$, it can be shown that the closed loop

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[1] The controller is said to be a proportional controller because $c(t) \propto e(t)$. More sophisticated servo control systems tend to use a combination of proportional integral and derivative (PID) control. In that case, $c(t) = K_\mathrm{P} e(t) + K_\mathrm{I} \int_{-\infty}^{t} e(\tau)\, d\tau + K_\mathrm{D} \frac{de(t)}{dt}$.

transfer function

$$T(s) = \frac{\Theta(s)}{\Theta_d(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)} = \frac{KA/\tau_m}{s^2 + (1/\tau_m)s + KA/\tau_m}. \tag{4}$$

This is a second order system and it can be written in the standard form

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{5}$$

where $\omega_n = \sqrt{\frac{KA}{\tau_m}}$ and $\zeta = \frac{1}{2\omega_n \tau_m}$. If the input $\theta_d(t)$ is a unit step function, then the output is the step response,

$$\theta_{\text{step}}(t) = \mathscr{L}_t^{-1}\left\{\frac{T(s)}{s}\right\}, \tag{6}$$

where $x(t) = \mathscr{L}_t^{-1}\{X(s)\}$ denotes the inverse Laplace Transform of $X(s)$ written as a function of t. Now assume that K is chosen so that $T(s)$ is under-damped. That is, K is chosen such that $s^2 + (1/\tau_m)s + KA/\tau_m$ has complex roots. Equivalently, K is chosen such that $\zeta < 1$. In that case, we have that for $t \geq 0$,

$$\theta_{\text{step}}(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \phi\right), \tag{7}$$

where $\phi = \operatorname{atan}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \operatorname{acos}(\zeta)$. A plot of a generic step response from an under-damped second-order system is given in Figure 2. A convenient value of K for closed-loop system identification is that which generates about 40% overshoot; that is the value of K for which

$$\max_t \theta_{\text{step}}(t) \approx 1.4 \lim_{t\to\infty} \theta_{\text{step}}(t). \tag{8}$$

If the closed loop is under-damped, the values of A and τ_m can be identified from the 'peak' of the frequency response of $T(s)$ in (5) in the following way: Observe that

$$|T(j\omega)|^2 = \frac{\omega_n^4}{|\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega|^2}. \tag{9}$$

Sketches of $|T(j\omega)|$ on a log-log scale for different values of ζ are provided in Figure 3. By differentiating the denominator with respect to ω and setting the derivative to zero, we see that the denominator reaches a minimum, and hence $|T(j\omega)|$ reaches a maximum, when $\omega = \omega_p$, where

$$\omega_p = \omega_n\sqrt{1 - 2\zeta^2}. \tag{10}$$

The value of the peak is

$$M_p^2 = \max_\omega |T(j\omega)|^2 = |T(j\omega_p)|^2 = \frac{1}{4\zeta^2(1-\zeta^2)}. \tag{11}$$

Once we have M_p we can solve (11) for ζ, and then we can solve (10) for ω_n. Once we have ζ and ω_n, we can find A and τ_m using the equations which follow (5).

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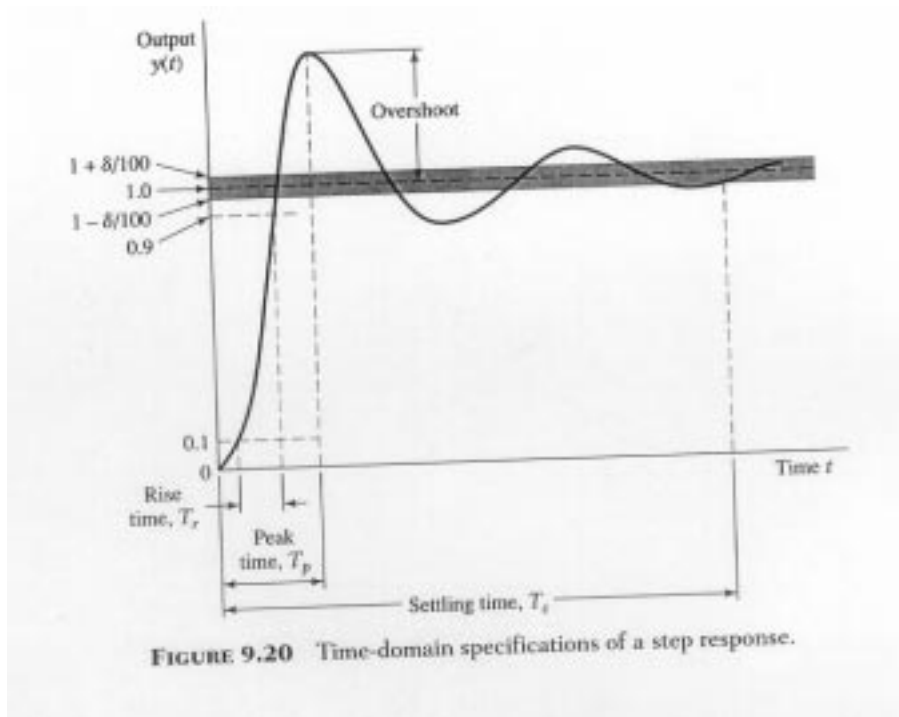FIGURE 9.20 Time-domain specifications of a step response.

Figure 2: Step response of an under-damped second-order system (Figure 9.20 of Haykin and Van Veen, *Signals and Systems*, Wiley, 1999).
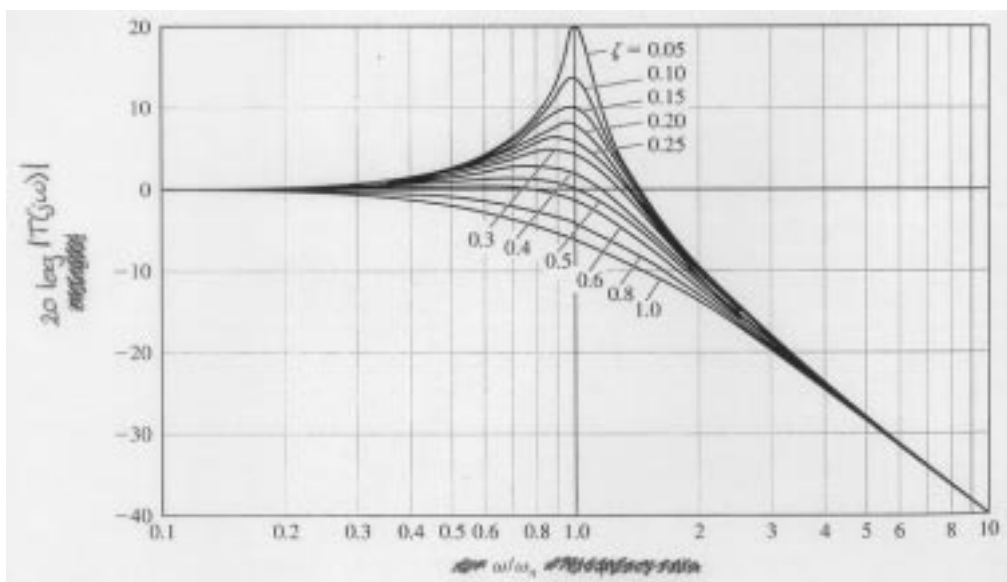

Figure 3: Bode plots of $|T(j\omega)|$ in (9) for various values of ζ with a normalized frequency axis (Figure 8.10 of Dorf and Bishop, *Modern Control Systems*, Ninth Edition, Prentice Hall, 2001).
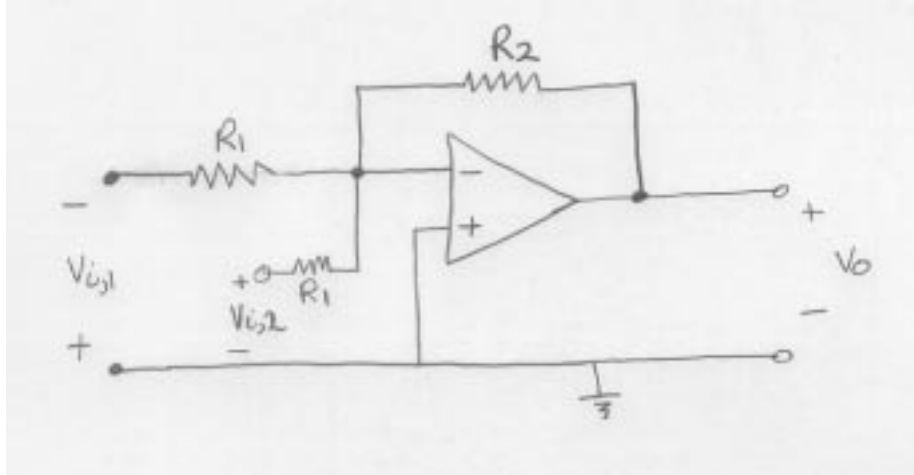
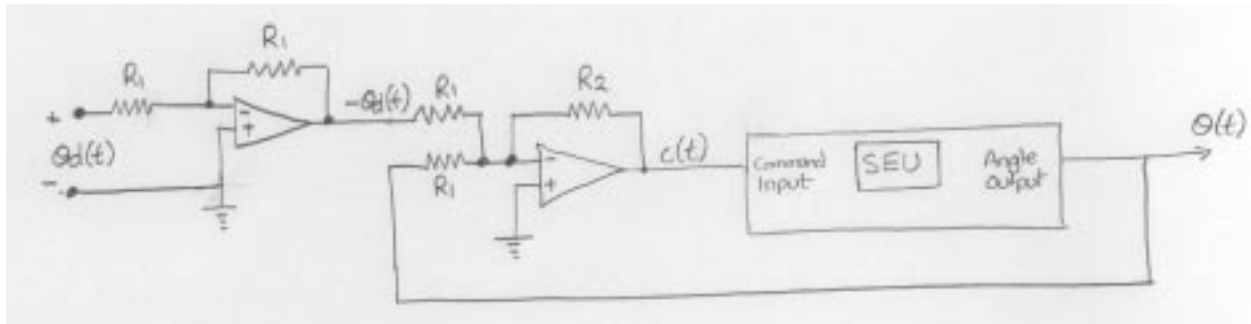Figure 4: Summing amplifier with one inverted input.

Figure 5: Closed loop circuit for experiments.

4 Perform Closed Loop Identification

We will perform the above closed loop identification using two steps. First we will observe that by choosing $K = 1$ in Figure 1 we obtain an appropriate overshoot. Then we will identify the peak frequency and peak gain.

Before we begin, we must determine how to construct $e(t)$ and $c(t)$ in practice. To do so, consider the circuit in Figure 4. Using nodal analysis and an assumption that the op-amp is ideal it can be shown that

$$V_o(s) = -\frac{R_2}{R_1}\big(V_{i,2}(s) - V_{i,1}(s)\big). \tag{12}$$

Therefore, this circuit can be used to implement the summer and the proportional gain in Figure 1, as shown in Figure 5.

4.1 Overshoot (2 marks)

To obtain the marks for this question, demonstrate the scope trace to your TA.

Use the following procedure to show that by choosing $K = 1$ in Figure 1 we obtain an overshoot that is appropriate for the identification of the motor parameters.

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i) Connect the ground of the op-amp module to the ground of the scope. Connect the ground of the signal generator (black) to the ground of the op-amp module and connect the ground of the op-amp module to the ground of the SEU.

ii) Connect the active (red) output of the signal generator to an 'isolated' terminal on the op-amp module. Switch on the signal generator, and set it to produce a 6V peak-to-peak square wave of frequency 0.1 Hz.

iii) Switch on the scope and set the vertical axis to 5V/div and the horizontal axis to 2s/div. Set the coupling to DC. Put the scope in 'roll mode' by pressing the $\boxed{\frac{\text{Main}}{\text{Delayed}}}$ button above the Time/div knob and choosing 'Roll' from the on-screen menu.

iv) Using the ring/banana plug converters, obtain a plot of your square wave on the scope. Confirm that it is indeed 6V peak-to-peak and 0.1 Hz.

v) Turn off the signal generator and construct the circuit in Figure 5 with $R_1 = R_2 = 10$ kΩ. Test the resistances of the orange components you have been given using the multi-meter as an Ohmmeter and verify that they have 10 kΩ resistance. You will need five resistors, four flat connectors and several cables to construct this circuit. (Do you really have to construct the inverter to produce $-\theta_d(t)$ for the purposes of this experiment?).

vi) Set the motor to inhibit. Set the resolution to 2 revs. Switch on the op-amp module, reset the module and then set to 'compute'. Turn on the signal generator. Set the motor to enable and observe the angle output on the scope. Observe that the overshoot is approximately 40%.

vii) Set the motor to inhibit.

4.2 Peak frequency and peak gain (2 marks)

To obtain your marks for this question you must show your TA the scope trace for input frequency ω_p and provide the corresponding values of A and τ_m.

Use the following procedure to determine the peak frequency and peak gain of your motor, and hence obtain A and τ_m.

i) Set the motor to inhibit.

ii) Set the signal generator to generate a sinusoidal signal of 6V peak-to-peak at frequency 1 Hz.

iii) Leave the scope in roll mode and set the vertical axis to 2V/div and the time axis to 1 sec/div. Display the generated signal on the scope and confirm that it is as required.

iv) Apply the generated signal to the circuit in Figure 5 and set the motor to enable. Display the angle output on the scope and measure the peak-to-peak output voltage $\theta(t)$. Hence calculate

$$|T(j2\pi f)|_{f=1} = \frac{\text{peak-to-peak output voltage}}{\text{peak-to-peak input voltage} = 6\text{V}}. \tag{13}$$

v) Decrease the input frequency using the dial, and observe the peak-to-peak output voltage of $\theta(t)$. Hence calculate $|T(j2\pi f)|$ for $f = 0.9, 0.7, 0.5, 0.3, 0.1$ Hz. Note that you may have to adjust the horizontal scale on the scope. Observe that at low frequencies some of the non-linearities of the motor become significant. (If the system were perfectly linear, the output would be a perfect sinusoid at the same frequency as the input.)

vi) Return the input frequency to 1 Hz. Increase the input frequency using the dial, and observe the peak-to-peak output voltage of $\theta(t)$. Hence calculate $|T(j2\pi f)|$ for $f = 2, 3, 5, 7, 10$ Hz. Note that you may have to adjust the horizontal scale on the scope. Observe that the output signal becomes quite noisy and that the some different non-linear effects become significant at higher frequencies.

vii) From the above measurements it should be clear that the peak of the frequency response lies somewhere between 1 Hz and 3 Hz. Set the horizontal axis of the scope to 200 msec/div and the vertical axis to 2V/div or 5V/div. *Slowly* increase the frequency of the signal generator from 1 Hz (using the dial) and search for the frequency at which the peak gain occurs. Try to obtain an accuracy of at least 0.05 Hz in your estimation of the peak frequency.

viii) Record your values of ω_p and $|T(j\omega_p)|$ and hence determine A and τ_m using the theory in Section 3.

ix) Set the motor to inhibit.

5   Trade-offs in the Design of a Proportional Controller

When designing a simple proportional controller for a servomechanism we often consider the following two performance criteria:

1. the 10% to 90% rise time, T_r

2. the percentage overshoot.

The step response of a typical under-damped second-order system was given in Figure 2, where the concepts of rise time and overshoot were implicitly defined. We would like to make the rise time small while simultaneously keeping the overshoot small. Unfortunately these are conflicting objectives, as we will now show.[2]

To determine that our performance criteria conflict, we observe from Example 9.5 in Haykin and Van Veen, *Signals and Systems*, Wiley, 1999, that for a general second order system of the form in (5) the percentage overshoot is given by

$$\text{percentage overshoot} = 100 \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right). \tag{14}$$

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[2]This is one of the reasons why practical controllers for servomechanisms tend to involve some form of integral and derivative action, as well as just proportional action.

It can also be shown that for $0.3 \leq \zeta \leq 0.8$,

$$T_r \approx \frac{2.16\zeta + 0.6}{\omega_n}. \tag{15}$$

For our servomechanism, $\omega_n = \sqrt{\frac{KA}{\tau_m}}$ and $\zeta = \frac{1}{2\omega_n \tau_m}$. Hence

$$\text{percentage overshoot} = 100 \exp\left(-\frac{\pi}{\sqrt{4KA\tau_m - 1}}\right) \tag{16}$$

and

$$T_r \approx \frac{2.16 + 1.2\sqrt{KA\tau_m}}{4KA}. \tag{17}$$

Therefore, to reduce the rise time we must increase K, but this will increase the overshoot.

6 Observe the rise-time/overshoot trade-off (2 marks)

To obtain your marks for this question you must show your TA the scope trace for several values of R_2.

Observe the rise-time/overshoot trade-off using the following procedure:

i) Set the motor to inhibit.

ii) Switch off the op-amp module.

iii) Replace the resistor R_2 by the 100 kΩ potentiometer. Connect one lead to a yellow terminal and the other to the black 'tap'.

iv) Set up the signal generator and scope in the same way as in items ii) and iii) of Section 4.1.

v) Turn on the op-amp module, reset it and the set it to 'compute'.

vi) Set the motor to enable.

vii) Adjust the resistance of the potentiometer and observe that by increasing R_2 (i.e., increasing K), we reduce the rise time but increase the overshoot.

viii) Set the motor to inhibit.

7 Design of a Proportional Servo Controller

In practice, one would typically specify a maximum tolerable overshoot and employ the largest K which satisfies that constraint. The task of this section is to design a proportional controller for your servomechanism which generates no more than 25% overshoot.

7.1 Computation of required gain (2 mark)

To obtain your marks for this section, show your TA your computation of R_2 and the value you obtained.

Recall that $K = 1$ generates 40% overshoot. If we specify a maximum tolerable overshoot of 25% then K must be reduced. Hence the value of the resistor R_2 in Figure 5 must be less than R_1. Use (16) to determine the value of K required. Hence determine the value of R_2 when $R_1 = 100$ kΩ.

7.2 Implementation and verification (2 mark)

To obtain your marks for this section your must show your TA that the overshoot is less than 25%. You must also compare the value of R_2 which you implemented to that which you calculated.

Now verify that your design does indeed have less than 25% overshoot via the following procedure:

i) Set the motor to inhibit and Turn off the op-amp module.

ii) Remove the inverting amplifier on the left of Figure 5 and apply the signal generator to the point marked "$-\theta_d(t)$".

iii) Replace the current resistors marked R_1 by 100 kΩ resistors (yellow).

iv) Disconnect the potentiometer from the circuit at connect it to the multi-meter configured as an Ohmmeter. Set its resistance to the desired value.

v) Disconnect the multi-meter and connect the potentiometer as R_2.

vi) Switch on the op-amp module, reset it, then set it to 'compute'. Leave the signal generator in its previous configuration and set the motor to enable.

vii) Measure the overshoot.

viii) How accurate was your design? Did you achieve an overshoot of less than 25%? If not, adjust the value of R_2 (online) until the desired overshoot is obtained. Then remove the potentiometer from the circuit and measure its resistance. How far off were you?