Solutions For Midterm Exam.

Q1

Classify each of the following systems in terms of Linearit[y] and time invariance. Justify your answers.

(a) $y''(t) + 3y'(t) = 2x'(t) + x(t)$

This is a standard representation of LTI syste[m] by differential equation.

So, Linear & Time invariant.

(b) $y''(t) + 2y(t)y'(t) = x(t)$

Linearity: The differential equation is nonlinear So, we can expect the system to be non linea[r]

For linearity $\quad x(t) \rightarrow y(t) \xRightarrow[\text{(one Condition)}]{\text{linearity}} \alpha x(t) \rightarrow \alpha y$

$\alpha \neq 0$

If $\alpha x(t) \rightarrow \alpha y(t)$ Then

$$\alpha y''(t) + 2\alpha y(t) \cdot \alpha y'(t) = \alpha x(t)$$

$$\Rightarrow \quad y''(t) + 2\alpha y(t) y'(t) = x(t)$$

which is not the original diff. eq.

Thus $y(t)$ is not the solution to original eq.

$\Rightarrow$ system is non linear (

For time-invariance

If the initial conditions are zero

$\Rightarrow$ the system is time-invariant because we do not have "t" out side "y" and "x" arguments

(C)

$$y[K+1] + y[K] = K\,x[K]$$

The system is linear

Assume $y_1[K+1] + y_1[K] = K x_1[K]$

$y_2[K+1] + y_2[K] = K x_2[K]$

define: $y_3[K] := a y_1[K] + b y_2[K]$

$x_3[K] := a x_1[K] + b x_2[K]$

Let see if $y_3[K+1] + y_3[K] \overset{?}{=} K x_3[K]$

$$a y_1[K+1] + b y_2[K+1] + a y_1[K] + b y_2[K]$$

$$\overset{?}{=} K a x_1[K] + K b x_2[K]$$

LHS: $a\left(\underbrace{y_1[K+1] + y_1[K]}_{K x_1[K]}\right) + b\left(\underbrace{y_2(K+1) + y_2[K]}_{K x_2[K]}\right)$

$$\overset{\checkmark}{=} K a x_1[K] + K b x_2[K]$$

The system is tim variant because k is out $x[\cdot]$ on the RHS of eq.

(d) $y[k+2] - y[k+1] = x[k]$

standard LTIC representation with difference equation.

Q2. Consider the discrete-time system

$$y[k] - \frac{3}{4} y[k-1] + \frac{1}{8} y[k-2] = 2x[k]$$

with the initial Conditions $y[-1] = 1$ and $y[-2]$

and excitation is given by $x[k] = 2u[k]$.

(a) Find the response of the system due to its initial Condi

(b) Find the total response of the system

(C) What are the natural and forced responses of the system.

(a) Characteristic equation:

$$\gamma^2 - \frac{3}{4}\gamma + \frac{1}{8} = 0$$

$$(\gamma - \tfrac{1}{2})(\gamma - \tfrac{1}{4}) = 0 \Rightarrow \gamma_1 = \tfrac{1}{2} \ \& \ \gamma_2 = \tfrac{1}{4}$$

The zero input response is given by:

$$y_0[k] = C_1 (\tfrac{1}{2})^k + C_2 (\tfrac{1}{4})^k$$

Applying the initial conditions we get:

$(K=-1)$ $\quad C_1\left(\frac{1}{2}\right)^{-1} + C_2\left(\frac{1}{4}\right)^{-1} = 1 \Rightarrow 2C_1 + 4C_2 = 1$

$(K=-2)$ $\quad C_1\left(\frac{1}{2}\right)^{-2} + C_2\left(\frac{1}{4}\right)^{-2} = -1 \Rightarrow 4C_1 + 16C_2 = -1$

$$\Rightarrow \begin{cases} 4C_1 + 8C_2 = 2 \\ -4C_1 - 16C_2 = 1 \end{cases} \Rightarrow \begin{cases} C_1 = 5/4 \\ C_2 = -3/8 \end{cases}$$

Thus:

$$\boxed{y_0[K] = \frac{5}{4}\left(\frac{1}{2}\right)^K - \frac{3}{8}\left(\frac{1}{4}\right)^K}$$

(b) To find the total response of the system since we have already obtained Z.I.R., we need to find Z.S.R.

First we have to find $h[K]$ (impulse respon

To do so, we need initial conditions

$$K=0, \quad h[0] - \frac{3}{4}\cancelto{0}{h[-1]} + \frac{1}{8}\cancelto{0}{h[-2]} = 2\cancelto{1}{\delta[0]}$$

$$\Rightarrow h[0] = 2$$

$$K=1, \quad h[1] - \frac{3}{4}\cancelto{2}{h[0]} + \frac{1}{8}\cancelto{0}{h[-1]} = 2\cancelto{0}{\delta[1]}$$

$$\Rightarrow h[1] = 3/2$$

$$h[K] = \cancelto{0}{A_0}\,\delta[K] + \left[B_1\left(\frac{1}{2}\right)^K + B_2\left(\frac{1}{4}\right)^K\right]u[K]$$

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$h[0] = B_1 + B_2 = 2$

$h[1] = \frac{1}{2}B_1 + \frac{1}{4}B_2 = 3/2$ $\Rightarrow B_1 = 4,\ B_2 = -2$

$$\Rightarrow h[K] = 4\left(\tfrac{1}{2}\right)^K u[K] - 2\left(\tfrac{1}{4}\right)^K u[K]$$

- Now the Z.S.R. is given by:

$$y_{zs}[K] = h[K] * x[K]$$

$$\Rightarrow y_{zs}[K] = 2\,u[K] * \left[4\left(\tfrac{1}{2}\right)^K u[K] - 2\left(\tfrac{1}{4}\right)^K u[K]\right]$$

Using the convolution table we get:

$$y_{zs}[K] = \left[\frac{32}{3} - 16\left(\tfrac{1}{2}\right)^{K+1} + \frac{16}{3}\left(\tfrac{1}{4}\right)^{K+1}\right] u[K]$$

- Total response : $y[K] = y_0[K] + y_{zs}[K]$

$$\Rightarrow \boxed{y[K] = \frac{5}{4}\left(\tfrac{1}{2}\right)^K - \frac{3}{8}\left(\tfrac{1}{4}\right)^K + \left[\frac{32}{3} - 16\left(\tfrac{1}{2}\right)^{K+1} + \frac{16}{3}\left(\tfrac{1}{4}\right)^{K+1}\right]u[K]}$$

(C) Forced response is :

$$\boxed{y_\phi[K] = \frac{32}{3}\,u[K]}$$

Natural response is :

$$\boxed{\left(\frac{5}{4} - 8\,u[K]\right)\left(\tfrac{1}{2}\right)^K + \left(-\frac{3}{8} + \frac{4}{3}u[K]\right)\left(\tfrac{1}{4}\right)^K}$$

Q3 Consider the continous-time system $y''(t)+6y'(t)+8y(t)=2x(t)$ with the initial conditions $y(0^-)=1$ and $y'(0^-)=1$ and excitation $x(t)=e^{-t}u(t)$.

(a) What is the zero input response of the system?

(b) Find the unit impulse response of the system.

(c) Find the total response of the system.

(a) Z.I.R ?

The characteristic eq. is given by:

$$\lambda^2+6\lambda+8=0$$

$$(\lambda+4)(\lambda+2)=0 \Rightarrow \lambda_1=-4, \lambda_2=-2$$

$\Rightarrow$ Z.I.R is $y_0(t)=C_1e^{-4t}+C_2e^{-2t}$

Applying the initial conditions:

$$-1=C_1+C_2$$

$$1=-4C_1-2C_2$$

$$\Rightarrow \boxed{y_0(t)=\frac{1}{2}e^{-4t}-\frac{3}{2}e^{-2t}}$$

(b) To get unit impulse response of the system

$$h(t)=b_n\delta(t)+[P(D)y_n(t)]u(t)$$

For our problem $b_n=b_2=0$

$$\Rightarrow h(t)=P(D)y_n(t)u(t)$$

$$=2y_n(t)u(t)$$

The initial conditions are:

$$y_n'(0) = 1 \quad \text{and} \quad y_n(0) = 0$$

Then $\quad y_n(t) = C_3 e^{-4t} + C_4 e^{-2t}$

Applying initial conditions: $C_3 = -\frac{1}{2}$, $C_4 = \frac{1}{2}$

$$\Rightarrow h(t) = 2\left(-\frac{1}{2} e^{-4t} + \frac{1}{2} e^{-2t}\right) u(t)$$

Thus : $\boxed{h(t) = \left(e^{-2t} - e^{-4t}\right) u(t)}$

c) The Z.S.R is thus given by:

$$y_{zs}(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \left(e^{-2\tau} - e^{-4\tau}\right) u(\tau) \cdot e^{-(t-\tau)}\, u(t-\tau)\, d\tau$$

$$= \int_{0}^{t} \left(e^{-2\tau} - e^{-4\tau}\right) e^{-(t-\tau)}\, d\tau$$

After integration

$$y_{zs}(t) = \left(\frac{2}{3} e^{-t} - e^{-2t} + \frac{1}{3} e^{-4t}\right) u(t)$$

- Total response

$y(t) = y_{zs}(t) + y_{zI}(t)$

$$\boxed{\Rightarrow y(t) = \left(\frac{2}{3} e^{-t} - e^{-2t} + \frac{1}{3} e^{-4t}\right) u(t) + \frac{1}{2} e^{-4t} - \frac{3}{2} e^{-2t}}$$

Q4 A Continous-time signal $x(t) = \cos(10^4 \pi t)$ is sampled at sampling frequencies $f_{s1} = 2.0$ KHz and $f_{s2} = 5.0$ KHz
For both cases answer the following questions:

(a) What are the corresponding discrete-time frequencies
Are the corresponding discrete time signals $x_1[k]$ and $x_2[k]$ Periodic? If Periodic, what is the Period for each case?

(b) Did aliasing occur? If so, what are the aliased frequencies?

(c) What is largest value of the sampling interval T to avoid aliasing?

(a) $\omega = 10^4 \pi$ rad/sec

$f = 5000$ Hz $= 5$ KHz

$T_1 = \frac{1}{2\,KHz} = 0.5$ msec.

$T_2 = \frac{1}{5\,KHz} = 0.2$ msec.

- The corresponding discrete-time frequencies:

$$\Omega_1 = \omega T_1 = 10^4 \pi \times 0.5 \times 10^{-3} = 5\pi$$

$$\Omega_2 = \omega T_2 = 2\pi$$

- Both D.T. signals are Periodic because

$$\Omega_1 = 5\pi = \frac{5}{2} \cdot 2\pi \quad \text{also}$$

$$\Omega_2 = 2\pi = \frac{1}{1} \cdot 2\pi$$

- The Periods are obtained through

$$N_{01} \cdot \Omega_1 = 2\pi \cdot m_1 \Rightarrow N_{01} = \frac{2\pi}{\Omega_1} \cdot m_1$$

$\Rightarrow N_{01} = \frac{2\pi}{5\pi} \cdot m_1 \Rightarrow m_1 = 5$ & $N_{01} = 2$

Similarly, $N_{02} = \frac{2\pi}{\Omega_2} \cdot m_2 \Rightarrow m_2 = 1$ & $N_{02} =$

(b) *Aliasing occured because*

$$\omega > \frac{\omega_{s1}}{2} \quad \& \quad \omega > \frac{\omega_{s2}}{2}$$

$$F_1 = \left| 5\,KHz - 2\,kHz \cdot m_1 \right|$$

$$\Rightarrow F_1 = 1\ kHz$$

$$F_2 = \left| 5\,KHz - 5\,KHz \cdot m_2 \right|$$

$$= 0\ Hz.$$

(C) *To avoid aliasing*

$$F_s \geq 10\,KHz \Rightarrow T \leq 0.1\ msec.$$

Q5 The everlasting signal $x[k] = 2\left(\frac{1}{2}\right)^k + 3\left(\frac{1}{5}\right)^k$ is applied to the discrete-time system $y[k] - y[k-1] = 2$
What is the output of the system?

$$H(r) = \frac{P(r)}{Q(r)} \Rightarrow H(r) = \frac{2}{r-1}$$

$$y[k] = 2 \cdot H\left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^k + 3 \cdot H\left(\frac{1}{5}\right) \cdot \left(\frac{1}{5}\right)^k$$

$$H\left(\frac{1}{2}\right) = \frac{2}{\frac{1}{2} - 1} = -4$$

$$H\left(\frac{1}{5}\right) = \frac{2}{\frac{1}{5} - 1} = -\frac{5}{2}$$

$$\Rightarrow y[k] = 2 \cdot (-4) \cdot \left(\frac{1}{2}\right)^k + 3 \cdot \left(-\frac{5}{2}\right)\left(\frac{1}{5}\right)^k$$

$$\Rightarrow \boxed{y[k] = -2\left(\frac{1}{2}\right)^k - \frac{15}{2}\left(\frac{1}{5}\right)^k}$$

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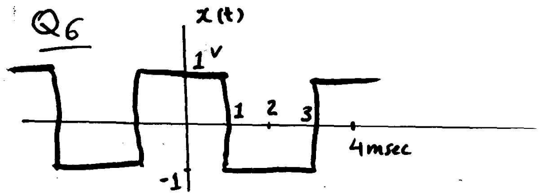
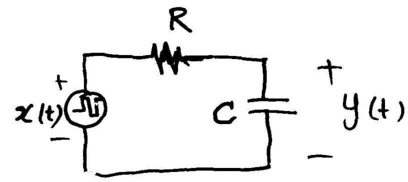
Q6





The Periodic signal $x(t)$ is applied to the shown Circuit. The time constant of the Circuit is $\tau = RC$ sec.

(a) Find the Complex Fourier series representation of signal x

(b) What is the complex Fourier series of the output $y(t)$ as a function of τ?

(c) What is the trigonometric Fourier Series of the output as a function of τ?

(d) If τ is a large value, what is approximately the output y of the circuit? Justify your answer.

(a) The signal is real, even, with half wave symmetry

$$\Rightarrow a_n = \frac{8}{T_0}\int_0^{T_0/4} 1.\cos n\omega_0 t \, dt$$

$$= \frac{8}{T_0}\cdot\frac{1}{n\omega_0}\sin n\omega_0 t \Big|_0^{T_0/4}$$

$$\Rightarrow a_n = \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

$$D_n = \frac{1}{2}a_n = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$\Rightarrow \boxed{x(t) = \sum_{n=-\infty}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\, e^{jn\omega_0 t}}$$

Or:

$$D_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\left[\int_0^{T_0/4} e^{-jn\omega_0 t} dt - \int_{T_0/4}^{3T_0/4} e^{-jn\omega_0 t} dt + \int_{\frac{3T_0}{4}}^{T_0} e^{-jn\omega_0 t} dt\right]$$

$$= \frac{1}{T_0}\left[\frac{-1}{jn\omega_0}\left(e^{-jn\omega_0 \frac{T_0}{4}} - 1\right) + \frac{1}{jn\omega_0}\left(e^{-jn\omega_0 \cdot \frac{3T_0}{4}} - e^{-jn\omega_0 \frac{T_0}{4}}\right) - \frac{1}{jn\omega_0}\left(e^{-jn\omega_0 T_0} - e^{-jn\omega_0 \cdot \frac{3T_0}{4}}\right)\right]$$

$$= \frac{1}{T_0}\cdot\frac{1}{jn\omega_0}\left[-e^{-jn\pi/2} + 1 + e^{-j3n\pi/2} - e^{-jn\pi/2} - \underbrace{e^{-jn2\pi}}_{=1} + e^{-j3n\pi/3}\right]$$

$$\underline{e^{-j3n\pi/2}} = e^{-j3n\pi/2} \cdot \underline{e^{j2\pi}}$$

$$= \frac{2}{j2\pi n}\left[e^{jn\pi/2} - e^{-jn\pi/2}\right] = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$\Rightarrow \boxed{x(t) = \sum_{n=-\infty}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\, e^{jn\omega_0 t}}$$

$$\underline{e^{+jn\pi/2}}$$

(b) $H(\omega) = \dfrac{1}{1+j\omega RC} \Rightarrow \dfrac{1}{1+j\omega\tau}$

$D'_n = D_n H(n\omega_0) \Rightarrow D'_n = \dfrac{2}{n\pi}\sin\dfrac{n\pi}{2} \cdot \dfrac{1}{1+jn\omega_0\tau}$

$$\Rightarrow \boxed{y(t) = \sum_{n=-\infty}^{\infty} \left[\frac{2}{n\pi}\,\frac{\sin n\pi/2}{1+jn\omega_0\tau}\right] e^{jn\omega_0 t}}$$

(c) $D'_n = \dfrac{2}{n\pi}\sin\dfrac{n\pi}{2}\,\dfrac{(1-jn\omega_0\tau)}{1+n^2\omega_0^2\tau^2}$

$\Rightarrow a'_n = \dfrac{4}{n\pi}\sin\dfrac{n\pi}{2} \cdot \dfrac{1}{1+n^2\omega_0^2\tau^2}$

$b'_n = \dfrac{4}{n\pi}\sin\dfrac{n\pi}{2} \cdot \dfrac{-n\omega_0\tau}{1+n^2\omega_0^2\tau^2}$

$$\Rightarrow \boxed{y(t) = \sum_{n=1}^{\infty} a'_n \cos n\omega_0 t + b'_n \sin n\omega_0 t}$$

(d) If τ is large, only DC goes through. But as $D_0 = 0$, output is approximately equal to zero.