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REVIEW QUESTIONS

2.1 State the necessary and sufficient conditions for the minimum of a function $f(x)$.

2.2 Under what circumstances can the condition $df(x)/dx = 0$ not be used to find the minimum of the function $f(x)$?

2.3 Define the rth differential, $d^r f(\mathbf{X})$, of a multivariable function $f(\mathbf{X})$.

2.4 Write the Taylor's series expansion of a function $f(\mathbf{X})$.

2.5 State the necessary and sufficient conditions for the maximum of a multivariable function $f(\mathbf{X})$.

2.6 What is a quadratic form?

2.7 How do you test the positive, negative, or indefiniteness of a square matrix $[A]$?

2.8 Define a saddle point and indicate its significance.

2.9 State the various methods available for solving a multivariable optimization problem with equality constraints.

2.10 State the principle behind the method of constrained variation.

2.11 What is the Lagrange multiplier method?

2.12 What is the significance of Lagrange multipliers?

2.13 Convert an inequality constrained problem into an equivalent unconstrained problem.

2.14 State the Kuhn–Tucker conditions.

2.15 What is an active constraint?

2.16 Define a usable feasible direction.

2.17 What is a convex programming problem? What is its significance?

2.18 Answer whether each of the following quadratic forms is positive definite, negative definite, or neither.

(a) $f = x_1^2 - x_2^2$
(b) $f = 4x_1x_2$
(c) $f = x_1^2 + 2x_2^2$
(d) $f = -x_1^2 + 4x_1x_2 + 4x_2^2$
(e) $f = -x_1^2 + 4x_1x_2 - 9x_2^2 + 2x_1x_3 + 8x_2x_3 - 4x_3^2$

2.19 State whether each of the following functions is convex, concave, or neither.

(a) $f = -2x^2 + 8x + 4$
(b) $f = x^2 + 10x + 1$
(c) $f = x_1^2 - x_2^2$
(d) $f = -x_1^2 + 4x_1x_2$
(e) $f = e^{-x}, x > 0$
(f) $f = \sqrt{x}, x > 0$
(g) $f = x_1x_2$
(h) $f = (x_1 - 1)^2 + 10(x_2 - 2)^2$

2.20 Match the following equations and their characteristics.

(a) $f = 4x_1 - 3x_2 + 2$	Relative maximum at (1, 2)
(b) $f = (2x_1 - 2)^2 + (x_1 - 2)^2$	Saddle point at origin
(c) $f = -(x_1 - 1)^2 - (x_2 - 2)^2$	No minimum
(d) $f = x_1x_2$	Inflection point at origin
(e) $f = x^3$	Relative minimum at (1, 2)

PROBLEMS

2.1 A dc generator has an internal resistance R ohms and develops an open-circuit voltage of V volts (Fig. 2.10). Find the value of the load resis-

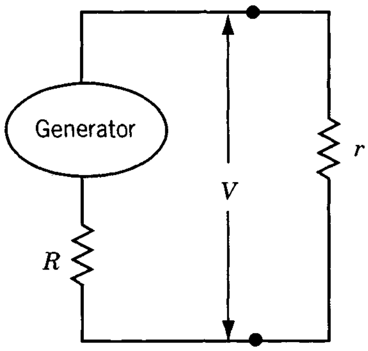

Figure 2.10 Electric generator with load.

tance r for which the power delivered by the generator will be a maximum.

2.2 Find the maxima and minima, if any, of the function

$$f(x) = \frac{x^4}{(x - 1)(x - 3)^3}$$

2.3 Find the maxima and minima, if any, of the function

$$f(x) = 4x^3 - 18x^2 + 27x - 7$$

2.4 The efficiency of a screw jack is given by

$$\eta = \frac{\tan \alpha}{\tan (\alpha + \phi)}$$

where α is the lead angle and ϕ is a constant. Prove that the efficiency of the screw jack will be maximum when $\alpha = 45° - \phi/2$ with $\eta_{\max} = (1 - \sin \phi)/(1 + \sin \phi)$.

2.5 Find the minimum of the function

$$f(x) = 10x^6 - 48x^5 + 15x^4 + 200x^3 - 120x^2 - 480x + 100$$

2.6 Find the angular orientation of a cannon to maximize the range of the projectile.

2.7 In a submarine telegraph cable the speed of signalling varies as $x^2 \log(1/x)$, where x is the ratio of the radius of the core to that of the covering. Show that the greatest speed is attained when this ratio is $1:\sqrt{e}$.

2.8 The horsepower generated by a Pelton wheel is proportional to $u(V - u)$, where u is the velocity of the wheel, which is variable, and V is the velocity of the jet, which is fixed. Show that the efficiency of the Pelton wheel will be maximum when $u = V/2$.

2.9 A pipe of length l and diameter D has at one end a nozzle of diameter d through which water is discharged from a reservoir. The level of water in the reservoir is maintained at a constant value h above the center of nozzle. Find the diameter of the nozzle so that the kinetic energy of the jet is a maximum. The kinetic energy of the jet can be expressed as

$$\frac{1}{4}\pi \rho d^2 \left(\frac{2gD^5h}{D^5 + 4fld^4}\right)^{3/2}$$

where ρ is the density of water, f the friction coefficient and g the gravitational constant.

2.10 An electric light is placed directly over the center of a circular plot of lawn 100 m in diameter. Assuming that the intensity of light varies directly as the sine of the angle at which it strikes an illuminated surface, and inversely as the square of its distance from the surface, how high should the light be hung in order that the intensity may be as great as possible at the circumference of the plot?

2.11 If a crank is at an angle θ from dead center with $\theta = \omega t$, where ω is the angular velocity and t is time, the distance of the piston from the end of its stroke (x) is given by

$$x = r(1 - \cos\theta) + \frac{r^2}{4l}(1 - \cos 2\theta)$$

where r is the length of the crank and l is the length of the connecting rod. For $r = 1$ and $l = 5$, find **(a)** the angular position of the crank at which the piston moves with maximum velocity, and **(b)** the distance of the piston from the end of its stroke at that instant.

Determine whether each of the following matrices is positive definite, negative definite, or indefinite by finding its eigenvalues.

2.12 $$[A] = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

2.13 $$[B] = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 4 & -2 \\ -4 & -2 & 4 \end{bmatrix}$$

2.14 $[C] = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -2 & -2 \\ -1 & -2 & -3 \end{bmatrix}$

Determine whether each of the following matrices is positive definite, negative definite, or indefinite by evaluating the signs of its submatrices.

2.15 $[A] = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

2.16 $[B] = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 4 & -2 \\ -4 & -2 & 4 \end{bmatrix}$

2.17 $[C] = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -2 & -2 \\ -1 & -2 & -3 \end{bmatrix}$

2.18 Express the function

$$f(x_1,x_2,x_3) = -x_1^2 - x_2^2 + 2x_1x_2 - x_3^2 + 6x_1x_3 + 4x_1 - 5x_3 + 2$$

in matrix form as

$$f(\mathbf{X}) = \tfrac{1}{2}\,\mathbf{X}^T\,[A]\,\mathbf{X} + \mathbf{B}^T\,\mathbf{X} + C$$

and determine whether the matrix $[A]$ is positive definite, negative definite, or indefinite.

2.19 Determine whether the following matrix is positive or negative definite.

$$[A] = \begin{bmatrix} 4 & -3 & 0 \\ -3 & 0 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

2.20 Determine whether the following matrix is positive definite.

$$[A] = \begin{bmatrix} -14 & 3 & 0 \\ 3 & -1 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

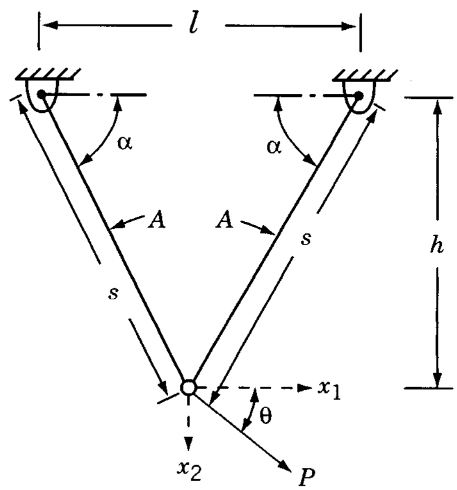

Figure 2.11 Two-bar truss.

2.21 The potential energy of the two-bar truss shown in Fig. 2.11 is given by

$$f(x_1, x_2) = \frac{EA}{s}\left(\frac{1}{2s}\right)^2 x_1^2 + \frac{EA}{s}\left(\frac{h}{s}\right)^2 x_2^2 - Px_1 \cos\theta - Px_2 \sin\theta$$

where E is Young's modulus, A the cross-sectional area of each member, l the span of the truss, s the length of each member, h the height of the truss, P the applied load, θ the angle at which the load is applied, and x_1 and x_2 are, respectively, the horizontal and vertical displacements of the free node. Find the values of x_1 and x_2 that minimize the potential energy when $E = 207 \times 10^9$ Pa, $A = 10^{-5}$ m^2, $l = 1.5$ m, $h = 4.0$ m, $P = 10^4$ N, and $\theta = 30°$.

2.22 The profit per acre of a farm is given by

$$20x_1 + 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2$$

where x_1 and x_2 denote, respectively, the labor cost and the fertilizer cost. Find the values of x_1 and x_2 to maximize the profit.

2.23 The temperatures measured at various points inside a heated wall are as follows:

Distance from the heated surface as a percentage of wall thickness, d	0	25	50	75	100
Temperature, t (°C)	380	200	100	20	0

It is decided to approximate this table by a linear equation (graph) of the form $t = a + bd$, where a and b are constants. Find the values of the constants a and b that minimize the sum of the squares of all differences between the graph values and the tabulated values.

2.24 Find the second-order Taylor's series approximation of the function

$$f(x_1,x_2) = (x_1 - 1)^2 e^{x_2} + x_1$$

at the points **(a)** (0,0) and **(b)** (1,1).

2.25 Find the third-order Taylor's series approximation of the function

$$f(x_1,x_2,x_3) = x_2^2 x_3 + x_1 e^{x_3}$$

at point (1,0,−2).

2.26 The volume of sales (f) of a product is found to be a function of the number of newspaper advertisements (x) and the number of minutes of television time (y) as

$$f = 12xy - x^2 - 3y^2$$

Each newspaper advertisement or each minute on television costs \$1000. How should the firm allocate \$48,000 between the two advertising media for maximizing its sales?

2.27 Find the value of x^* at which the following function attains its maximum:

$$f(x) = \frac{1}{10\sqrt{2\pi}} e^{-(1/2)[(x-100)/10]^2}$$

2.28 It is possible to establish the nature of stationary points of an objective function based on its quadratic approximation. For this, consider the quadratic approximation of a two-variable function as

$$f(\mathbf{X}) \approx a + \mathbf{b}^T\mathbf{X} + \tfrac{1}{2}\mathbf{X}^T[c]\,\mathbf{X}$$

where

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}, \quad \text{and} \quad [c] = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}$$

If the eigenvalues of the Hessian matrix, $[c]$, are denoted as β_1 and β_2, identify the nature of the contours of the objective function and the type of stationary point in each of the following situations.

(a) $\beta_1 = \beta_2$; both positive

(b) $\beta_1 > \beta_2$; both positive

(c) $|\beta_1| = |\beta_2|$; β_1 and β_2 have opposite signs

(d) $\beta_1 > 0, \beta_2 = 0$

Plot the contours of each of the following functions and identify the nature of its stationary point.

2.29 $f = 2 - x^2 - y^2 + 4xy$

2.30 $f = 2 + x^2 - y^2$

2.31 $f = xy$

2.32 $f = x^3 - 3xy^2$

2.33 Find the admissible and constrained variations at the point $\mathbf{X} = \begin{Bmatrix} 0 \\ 4 \end{Bmatrix}$ for the following problem:

$$\text{Minimize } f = x_1^2 + (x_2 - 1)^2$$

subject to

$$-2x_1^2 + x_2 = 4$$

2.34 Find the diameter of an open cylindrical can that will have the maximum volume for a given surface area, S.

2.35 A rectangular beam is to be cut from a circular log of radius r. Find the cross-sectional dimensions of the beam to **(a)** maximize the cross-sectional area of the beam, and **(b)** maximize the perimeter of the beam section.

2.36 Find the dimensions of a straight beam of circular cross section that can be cut from a conical log of height h and base radius r to maximize the volume of the beam.

2.37 The deflection of a rectangular beam is inversely proportional to the width and the cube of depth. Find the cross-sectional dimensions of a beam, which corresponds to minimum deflection, that can be cut from a cylindrical log of radius r.

2.38 A rectangular box of height a and width b is placed adjacent to a wall (Fig. 2.12). Find the length of the shortest ladder that can be made to lean against the wall.

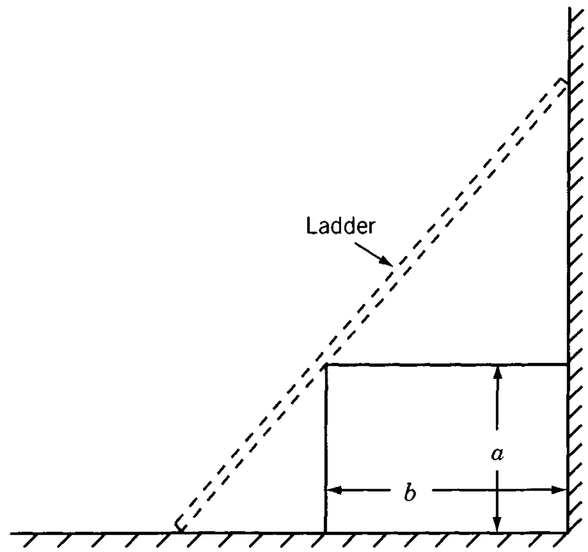

Figure 2.12 Ladder against a wall.

2.39 Show that the right circular cylinder of given surface (including the ends) and maximum volume is such that its height is equal to the diameter of the base.

2.40 Find the dimensions of a closed cylindrical soft drink can that can hold soft drink of volume V for which the surface area (including the top and bottom) is a minimum.

2.41 An open rectangular box is to be manufactured from a given amount of sheet metal (area S). Find the dimensions of the box to maximize the volume.

2.42 Find the dimensions of an open rectangular box of volume V for which the amount of material required for manufacture (surface area) is a minimum.

2.43 A rectangular sheet of metal with sides a and b has four equal square portions (of side d) removed at the corners, and the sides are then turned up so as to form an open rectangular box. Find the depth of the box that maximizes the volume.

2.44 Show that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to two-thirds of the diameter of the sphere. Also prove that the curved surface of the cone is a maximum for the same value of the altitude.

2.45 Prove Theorem 2.6.

2.46 A log of length l is in the form of a frustum of a cone whose ends have radii a and b $(a > b)$. It is required to cut from it a beam of uniform square section. Prove that the beam of greatest volume that can be cut has a length of $al/[3(a - b)]$.

2.47 It has been decided to leave a margin of 30 mm at the top and 20 mm each at the left side, right side, and the bottom on the printed page of a book. If the area of the page is specified as 5×10^4 mm^2, determine the dimensions of a page that provide the largest printed area.

2.48

$$\text{Minimize } f = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$$

subject to

$$x_1 + x_2 + 2x_3 = 3$$

by **(a)** direct substitution, **(b)** constrained variation, and **(c)** Lagrange multiplier method.

2.49

$$\text{Minimize } f(\mathbf{X}) = \tfrac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

subject to

$$g_1(\mathbf{X}) = x_1 - x_2 = 0$$

$$g_2(\mathbf{X}) = x_1 + x_2 + x_3 - 1 = 0$$

by **(a)** direct substitution, **(b)** constrained variation, and **(c)** Lagrange multiplier method.

2.50 Find the values of x, y, and z that maximize the function

$$f(x,y,z) = \frac{6xyz}{x + 2y + 2z}$$

when x, y, and z are restricted by the relation $xyz = 16$.

2.51 A tent on a square base of side $2a$ consists of four vertical sides of height b surmounted by a regular pyramid of height h. If the volume enclosed by the tent is V, show that the area of canvas in the tent can be expressed as

$$\frac{2V}{a} - \frac{8ah}{3} + 4a\sqrt{h^2 + a^2}$$

Also show that the least area of the canvas corresponding to a given volume V, if a and h can both vary, is given by

$$a = \frac{\sqrt{5}\, h}{2} \quad \text{and} \quad h = 2b$$

2.52 A departmental store plans to construct a one-story building with a rectangular planform. The building is required to have a floor area of 22,500 ft^2 and a height of 18 ft. It is proposed to use brick walls on three sides and a glass wall on the fourth side. Find the dimensions of the building to minimize the cost of construction of the walls and the roof assuming that the glass wall costs twice as much as that of the brick wall and the roof costs three times as much as that of the brick wall per unit area.

2.53 Find the dimensions of the rectangular building described in Problem 2.52 to minimize the heat loss assuming that the relative heat losses per unit surface area for the roof, brick wall, glass wall, and floor are in the proportion 4 : 2 : 5 : 1.

2.54 A funnel, in the form of a right circular cone, is to be constructed from a sheet metal. Find the dimensions of the funnel for minimum lateral surface area when the volume of the funnel is specified as 200 in^3.

2.55 Find the effect on f^* when the value of A_0 is changed to (**a**) 25π and (**b**) 22π in Example 2.10 using the property of the Lagrange multiplier.

2.56 (**a**) Find the dimensions of a rectangular box of volume $V = 1000\ \text{in}^3$ for which the total length of the 12 edges is a minimum using the Lagrange multiplier method.

(**b**) Find the change in the dimensions of the box when the volume is changed to 1200 in^3 by using the value of λ^* found in part (a).

(**c**) Compare the solution found in part (b) with the exact solution.

2.57 Find the effect on f^* of changing the constraint to (**a**) $x + x_2 + 2x_3 = 4$ and (**b**) $x + x_2 + 2x_3 = 2$ in Problem 2.48. Use the physical meaning of Lagrange multiplier in finding the solution.

2.58 A real estate company wants to construct a multistory apartment building on a 500 ft × 500 ft lot. It has been decided to have a total floor space of $8 \times 10^5\ \text{ft}^2$. The height of each story is required to be 12 ft, the maximum height of the building is to be restricted to 75 ft, and the parking area is required to be at least 10% of the total floor area according to the city zoning rules. If the cost of the building is estimated at \$$(500{,}000h + 2000F + 500P)$, where h is the height in feet, F is the floor area in square feet, and P is the parking area in square feet. Find the minimum cost design of the building.

2.59 Identify the optimum point among the given design vectors, $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$, by applying the Kuhn–Tlucker conditions to the following problem:

$$\text{Minimize } f(\mathbf{X}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

subject to

$$x_2^2 - x_1 \geq 0$$

$$x_1^2 - x_2 \geq 0$$

$$-\tfrac{1}{2} \leq x_1 \leq \tfrac{1}{2}, \qquad x_2 \leq 1$$

$$\mathbf{X}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \mathbf{X}_2 = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}, \quad \mathbf{X}_3 = \begin{Bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{Bmatrix}$$

2.60 Consider the following optimization problem:

$$\text{Maximize } f = -x_1^2 - x_2^2 + x_1 x_2 + 7x_1 + 4x_2$$

subject to

$$2x_1 + 3x_2 \leq 24$$

$$-5x_1 + 12x_2 \leq 24$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_2 \leq 4$$

Find a usable feasible direction at each of the following design vectors:

$$\mathbf{X}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \mathbf{X}_2 = \begin{Bmatrix} 6 \\ 4 \end{Bmatrix}$$

2.61 Consider the following problem:

$$\text{Minimize } f = (x_1 - 2)^2 + (x_2 - 1)^2$$

subject to

$$2 \geq x_1 + x_2$$

$$x_2 \geq x_1^2$$

Using Kuhn–Tucker conditions, find which of the following vectors are local minima:

$$\mathbf{X}_1 = \begin{Bmatrix} 1.5 \\ 0.5 \end{Bmatrix}, \quad \mathbf{X}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \mathbf{X}_3 = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

2.62 Using Kuhn–Tucker conditions, find the value(s) of β for which the point $x_1^* = 1, x_2^* = 2$ will be optimal to the problem:

$$\text{Maximize } f(x_1,x_2) = 2x_1 + \beta x_2$$

subject to

$$g_1(x_1,x_2) = x_1^2 + x_2^2 - 5 \le 0$$
$$g_2(x_1, x_2) = x_1 - x_2 - 2 \le 0$$

Verify your result using a graphical procedure.

2.63 Consider the following optimization problem:

$$\text{Maximize } f = -x_1 - x_2$$

subject to

$$x_1^2 + x_2 \ge 2$$
$$4 \le x_1 + 3x_2$$
$$x_1 + x_2^4 \le 30$$

(a) Find whether the design vector $\mathbf{X} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ satisfies the Kuhn–Tucker conditions for a constrained optimum.

(b) What are the values of the Lagrange multipliers at the given design vector?

2.64 Consider the following problem:

$$\text{Minimize } f(\mathbf{X}) = x_1^2 + x_2^2 + x_3^2$$

subject to

$$x_1 + x_2 + x_3 \ge 5$$
$$2 - x_2 x_3 \le 0$$
$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 2$$

Determine whether the Kuhn–Tucker conditions are satisfied at the following points:

$$\mathbf{X}_1 = \begin{Bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 2 \end{Bmatrix}, \quad \mathbf{X}_2 = \begin{Bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 3 \end{Bmatrix}, \quad \mathbf{X}_3 = \begin{Bmatrix} 2 \\ 1 \\ 2 \end{Bmatrix}$$

2.65 Find a usable and feasible direction **S** at (**a**) $\mathbf{X}_1 = \begin{Bmatrix} -1 \\ 5 \end{Bmatrix}$ and (**b**) $\mathbf{X}_2 = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$ for the following problem:

$$\text{Minimize } f(\mathbf{X}) = (x_1 - 1)^2 + (x_2 - 5)^2$$

subject to

$$g_1(\mathbf{X}) = -x_1^2 + x_2 - 4 \leq 0$$

$$g_2(\mathbf{X}) = -(x_1 - 2)^2 + x_2 - 3 \leq 0$$

2.66 Consider the following problem:

$$\text{Minimize } f = x_1^2 - x_2$$

subject to

$$26 \geq x_1^2 + x_2^2$$

$$x_1 + x_2 \geq 6$$

$$x_1 \geq 0$$

Determine whether the following search direction is usable, feasible, or both at the design vector $\mathbf{X} = \begin{Bmatrix} 5 \\ 1 \end{Bmatrix}$:

$$\mathbf{S} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \mathbf{S} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}, \quad \mathbf{S} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \mathbf{S} = \begin{Bmatrix} -1 \\ 2 \end{Bmatrix}$$

2.67 Consider the following problem:

$$\text{Minimize } f = x_1^3 - 6x_1^2 + 11x_1 + x_3$$

subject to

$$x_1^2 + x_2^2 - x_3^2 \le 0$$

$$4 - x_1^2 - x_2^2 - x_3^2 \le 0$$

$$x_i \ge 0, \quad i = 1,2,3, \quad x_3 \le 5$$

Determine whether the following vector represents an optimum solution:

$$\mathbf{X} = \begin{Bmatrix} 0 \\ \sqrt{2} \\ \sqrt{2} \end{Bmatrix}$$

2.68 $$\text{Minimize } f = x_1^2 + 2x_2^2 + 3x_3^2$$

subject to the constraints

$$g_1 = x_1 - x_2 - 2x_3 \le 12$$

$$g_2 = x_1 + 2x_2 - 3x_3 \le 8$$

using Kuhn–Tucker conditions.

2.69 $$\text{Minimize } f(x_1,x_2) = (x_1 - 1)^2 + (x_2 - 5)^2$$

subject to

$$-x_1^2 + x_2 \le 4$$
$$-(x_1 - 2)^2 + x_2 \le 3$$

by **(a)** the graphical method and **(b)** Kuhn–Tucker conditions.

2.70 $$\text{Maximize } f = 8x_1 + 4x_2 + x_1x_2 - x_1^2 - x_2^2$$

subject to

$$2x_1 + 3x_2 \le 24$$

$$-5x_1 + 12x_2 \le 24$$

$$x_2 \le 5$$

by applying Kuhn–Tucker conditions.

2.71 Consider the following problem:

$$\text{Maximize } f(x) = (x - 1)^2$$

subject to

$$-2 \leq x \leq 4$$

Determine whether the constraint qualification and Kuhn–Tucker conditions are satisfied at the optimum point.

2.72 Consider the following problem:

$$\text{Minimize } f = (x_1 - 1)^2 + (x_2 - 1)^2$$

subject to

$$2x_2 - (1 - x_1)^3 \leq 0$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Determine whether the constraint qualification and the Kuhn–Tucker conditions are satisfied at the optimum point.

2.73 Verify whether the following problem is convex:

$$\text{Minimize } f(\mathbf{X}) = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$$

subject to

$$2x_1 + x_2 \leq 6$$
$$x_1 - 4x_2 \leq 0$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

2.74 Check the convexity of the following problems.

(a) $$\text{Minimize } f(\mathbf{X}) = 2x_1 + 3x_2 - x_1^3 - 2x_2^2$$

subject to

$$x_1 + 3x_2 \leq 6$$
$$5x_1 + 2x_2 \leq 10$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

(b) $$\text{Minimize } f(\mathbf{X}) = 9x_1^2 - 18x_1x_2 + 13x_1 - 4$$

subject to

$$x_1^2 + x_2^2 + 2x_1 \geq 16$$