EE757
Numerical Techniques in Electromagnetics
Lecture 8

2D FDTD

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_x}\left(\frac{\partial H_z}{\partial y} - \sigma_x^e E_x - J_{ix}\right)$$

$$\frac{\partial E_y}{\partial t} = \frac{-1}{\varepsilon_y}\left(\frac{\partial H_z}{\partial x} + \sigma_y^e E_y + J_{iy}\right)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu_z}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

TE_z

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma_z^e E_z - J_{iz}\right)$$

$$\frac{\partial H_x}{\partial t} = -\frac{\partial E_z}{\mu_x \partial y}$$

$$\frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\mu_y \partial x}$$

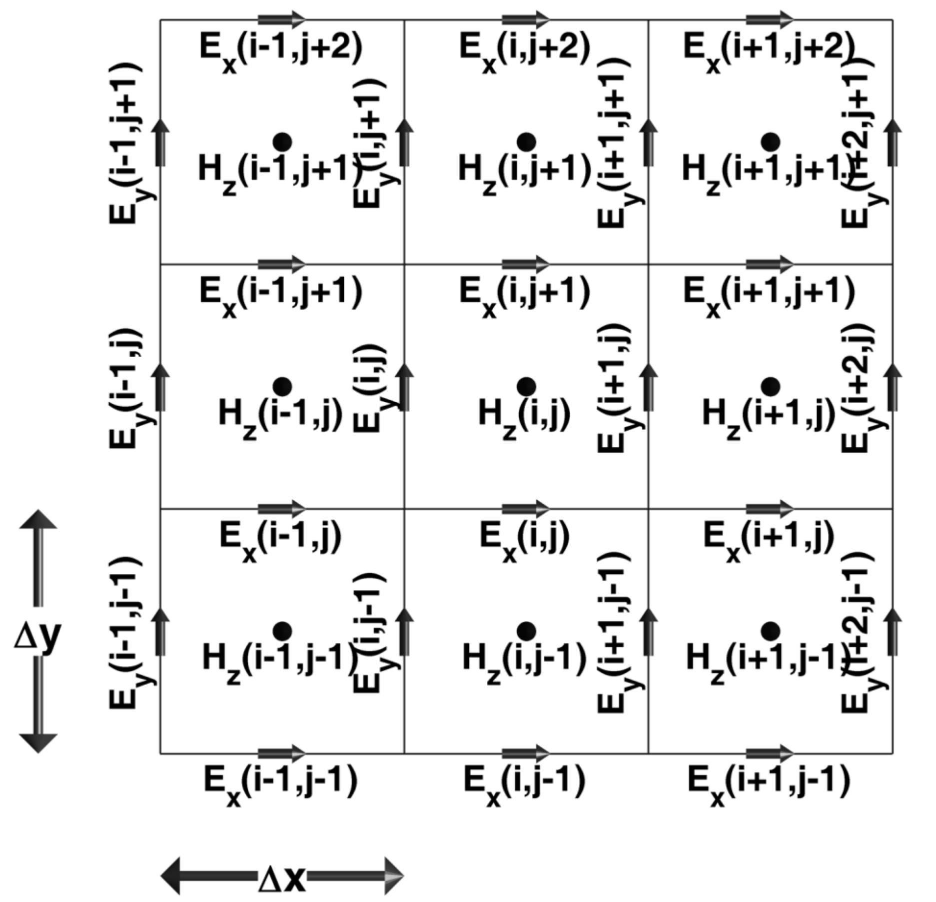
TM_z

TE_z Case

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_x}\left(\frac{\partial H_z}{\partial y} - \sigma_x^e E_x - J_{ix}\right)$$

$$\frac{\partial E_y}{\partial t} = \frac{-1}{\varepsilon_y}\left(\frac{\partial H_z}{\partial x} + \sigma_y^e E_y + J_{iy}\right)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu_z}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$



two electric field components and one magnetic component

$\mathbf{TE_z}$ Case (Cont'd)

$$E_x^{n+1}(i,j) = C_{exe}(i,j)\times E_x^{n}(i,j) + C_{exh}(i,j)\times\left(\frac{H_z^{n+\frac{1}{2}}(i,j) - H_z^{n+\frac{1}{2}}(i,j-1)}{\Delta y} - J_{ix}^{n+\frac{1}{2}}(i,j)\right)$$

$$E_y^{n+1}(i,j) = C_{eye}(i,j)\times E_y^{n}(i,j) + C_{eyh}(i,j)\times\left(-\frac{H_z^{n+\frac{1}{2}}(i,j) - H_z^{n+\frac{1}{2}}(i-1,j)}{\Delta x} - J_{iy}^{n+\frac{1}{2}}(i,j)\right)$$

$$H_z^{n+\frac{1}{2}}(i,j) = H_z^{n-\frac{1}{2}}(i,j) + C_{hze}(i,j)\times\left(\frac{E_x^{n}(i,j+1) - E_x^{n}(i,j)}{\Delta y} - \frac{E_y^{n}(i+1,j) - E_y^{n}(i,j)}{\Delta x}\right)$$

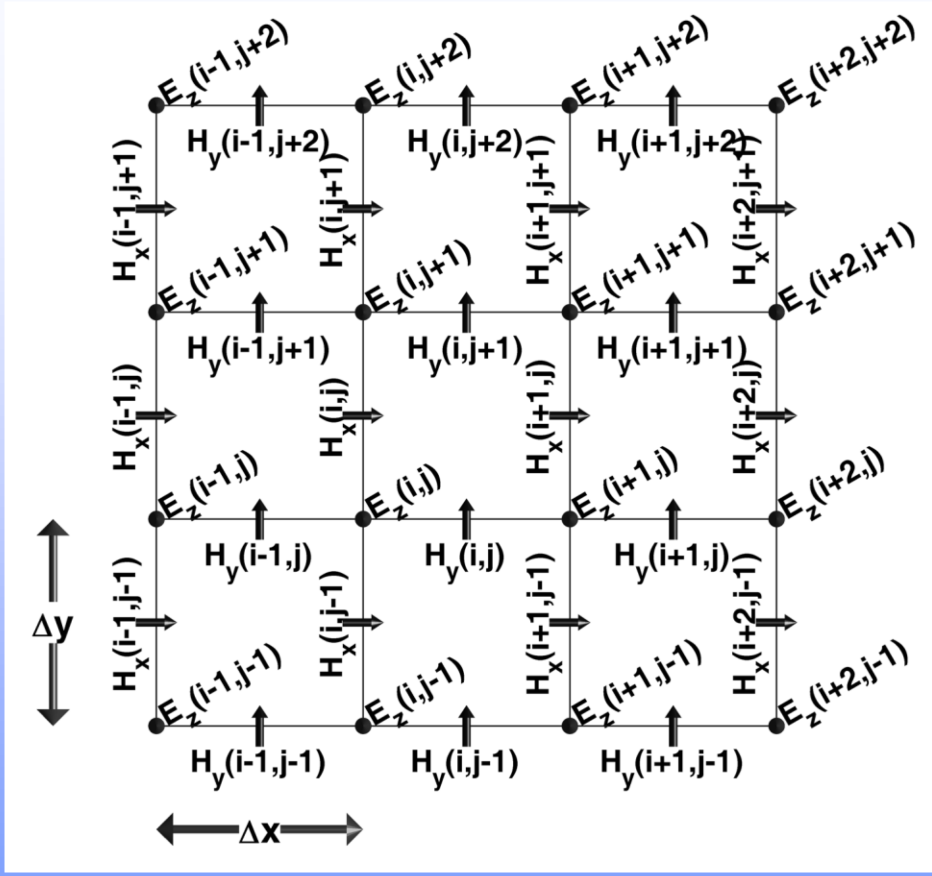
$(1/\Delta z) \rightarrow 0$ in 3D update equations

TM_z Case

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma_z^e E_z - J_{iz}\right)$$

$$\frac{\partial H_x}{\partial t} = -\frac{\partial E_z}{\mu_x \partial y}$$

$$\frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\mu_y \partial x}$$



TM_z Case (Cont'd)

$$E_z^{n+1}(i,j) = C_{eze}(i,j)\times E_z^{n}(i,j)$$
$$+\,C_{ezh}(i,j)\times\left(\frac{H_y^{n+\frac{1}{2}}(i,j)-H_y^{n+\frac{1}{2}}(i-1,j)}{\Delta x}-\frac{H_x^{n+\frac{1}{2}}(i,j)-H_x^{n+\frac{1}{2}}(i,j-1)}{\Delta y}-J_{iz}^{n+\frac{1}{2}}(i,j)\right)$$

$$H_x^{n+\frac{1}{2}}(i,j) = H_x^{n-\frac{1}{2}}(i,j)+C_{hxe}(i,j)\left(-\frac{E_z^{n}(i,j+1)+E_z^{n}(i,j)}{\Delta y}\right)$$

$$H_y^{n+\frac{1}{2}}(i,j) = H_y^{n-\frac{1}{2}}(i,j)+C_{hye}(i,j)\times\left(\frac{E_z^{n}(i+1,j)-E_z^{n}(i,j)}{\Delta x}\right),$$

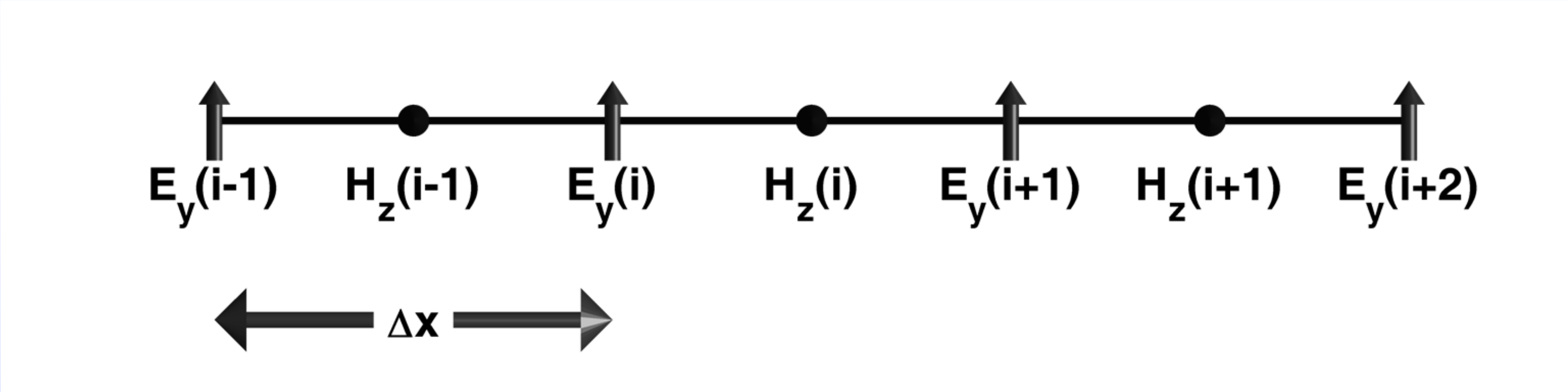
$(1/\Delta z)\rightarrow 0$ in 3D update equations

1D Maxwell's Equations

$$\frac{\partial E_y}{\partial t} = \frac{-1}{\varepsilon_y}\left(\frac{\partial H_z}{\partial x} + \sigma_y^e E_y + J_{iy}\right), \qquad \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\mu_z \partial x}. \qquad +\text{ve } x$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_z}\left(\frac{\partial H_y}{\partial x} - \sigma_z^e E_z - J_{iz}\right), \qquad \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\mu_y \partial x} \qquad -\text{ve } x$$

1D FDTD

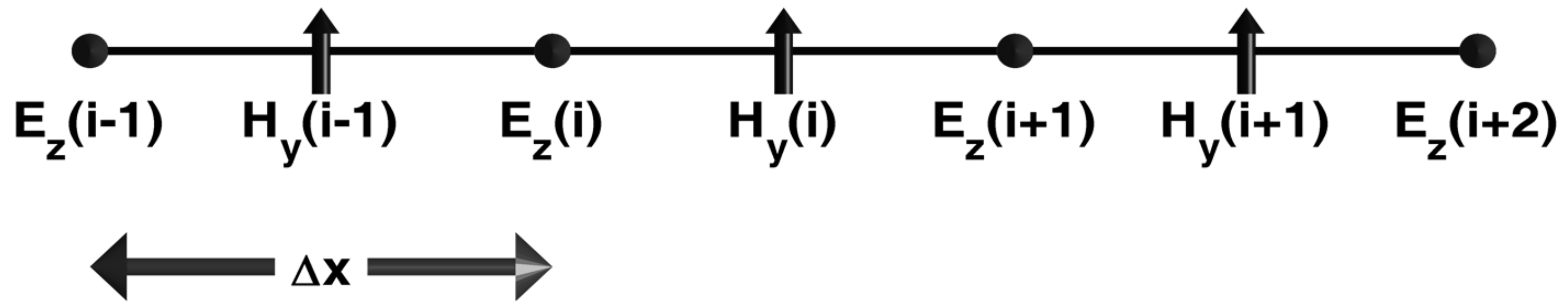


$$E_y^{n+1}(i) = C_{eye}(i)\times E_y^{n}(i) + C_{eyh}(i)\times\left(-\frac{H_z^{n+\frac{1}{2}}(i) - H_z^{n+\frac{1}{2}}(i-1)}{\Delta x} - J_{iy}^{n+\frac{1}{2}}(i)\right),$$

$$H_z^{n+\frac{1}{2}}(i) = H_z^{n-\frac{1}{2}}(i) + C_{hze}(i)\left(-\frac{E_y^{n}(i+1) - E_y^{n}(i)}{\Delta x}\right)$$

set $(1/\Delta y)\rightarrow 0$ and $(1/\Delta z)\rightarrow 0$ in 3D update equations

1D FDTD (cont'd)



$$E_z^{n+1}(i) = C_{eze}(i) \times E_z^n(i) + C_{ezh}(i) \times \left(\frac{H_y^{n+\frac{1}{2}}(i) - H_y^{n+\frac{1}{2}}(i-1)}{\Delta x} - J_{iz}^{n+\frac{1}{2}}(i) \right)$$

$$H_y^{n+\frac{1}{2}}(i) = H_y^{n-\frac{1}{2}}(i) + C_{hye}(i) \times \left(\frac{E_z^n(i+1) - E_z^n(i)}{\Delta x} \right)$$

set $(1/\Delta y) \rightarrow 0$ and $(1/\Delta z) \rightarrow 0$ in 3D update equations

The Courant-Friedrich-Levy (CFL) Limit

$$\Delta t \leq \frac{1}{c\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}}$$

if $\Delta x = \Delta y = \Delta z = \Delta h$, $\quad \Delta t \leq \frac{\Delta h}{c\sqrt{3}}$

the FDTD time-marching scheme becomes unstable if the time step exceeds the Courant limit

usually, we choose $\Delta t = 0.9$ CFL

CFL for 2D and 1D FDTD?

Boundary Conditions

- PEC
- PMC
- Absorbing Boundary Conditions

  Mur's First-order boundary condition

  Mur's Second-order boundary condition

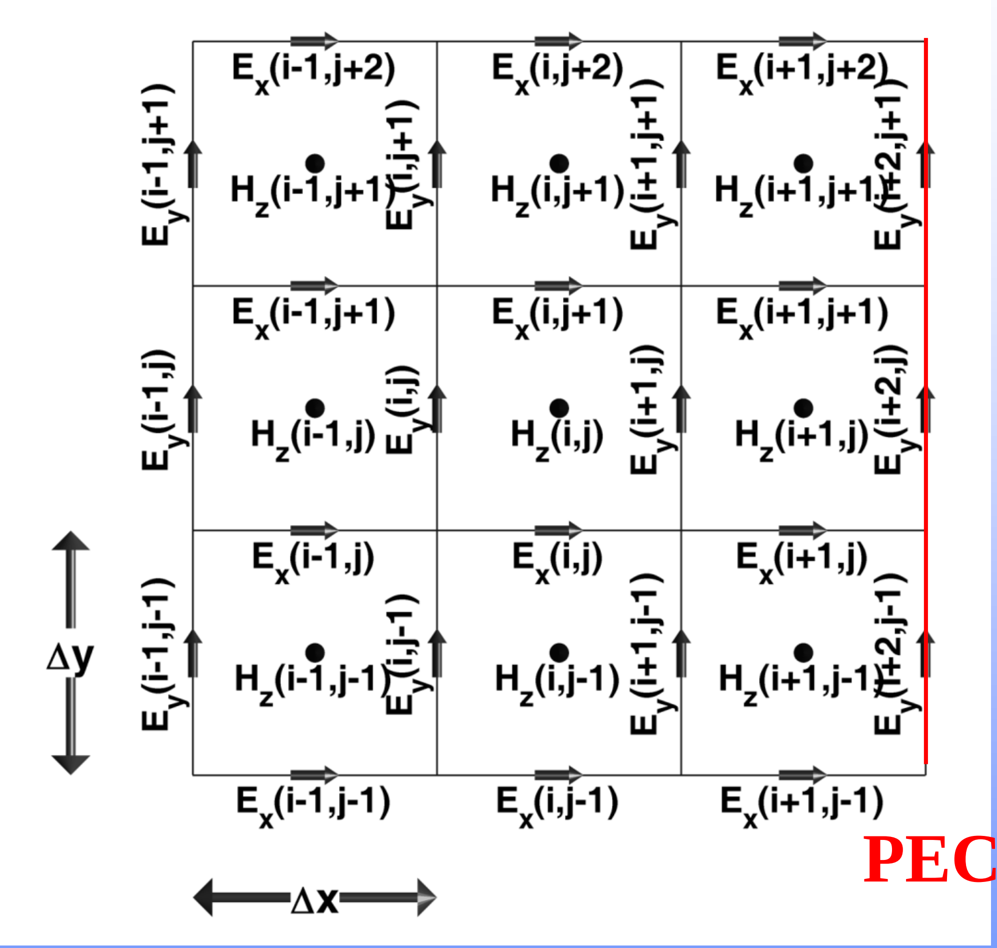
  Liao's boundary condition

  Introduction to PML

PEC: TE_z Case

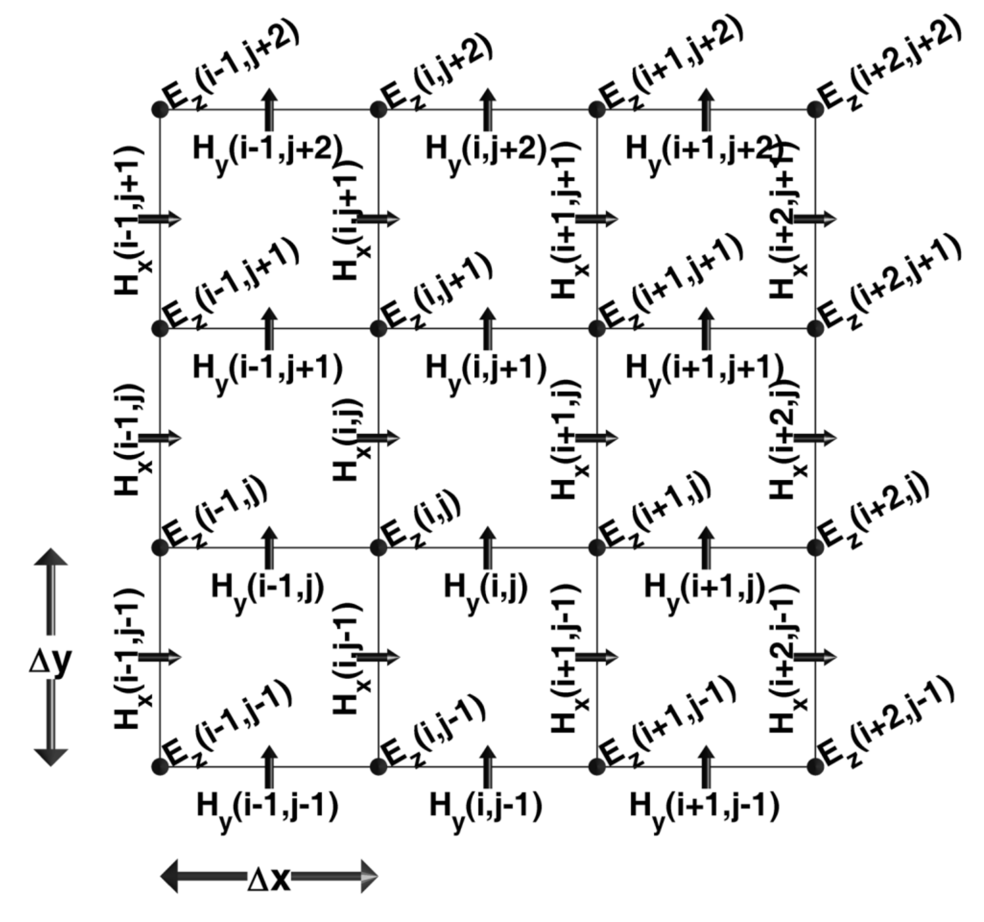
set all tangential E-field components at the boundary to zero for all time steps

FDTD update equations are applied only to interior electric and magnetic field components



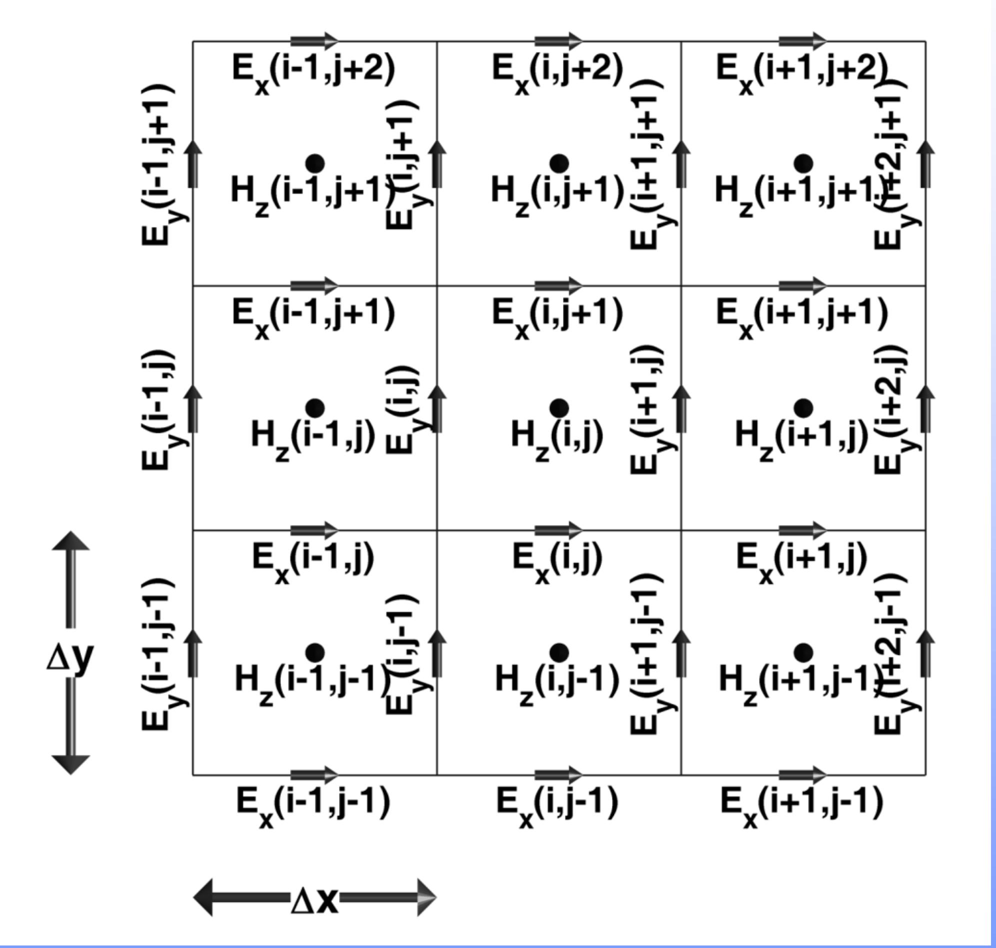
PEC: TM_z Case

set the E_z components at the electrical wall to zero



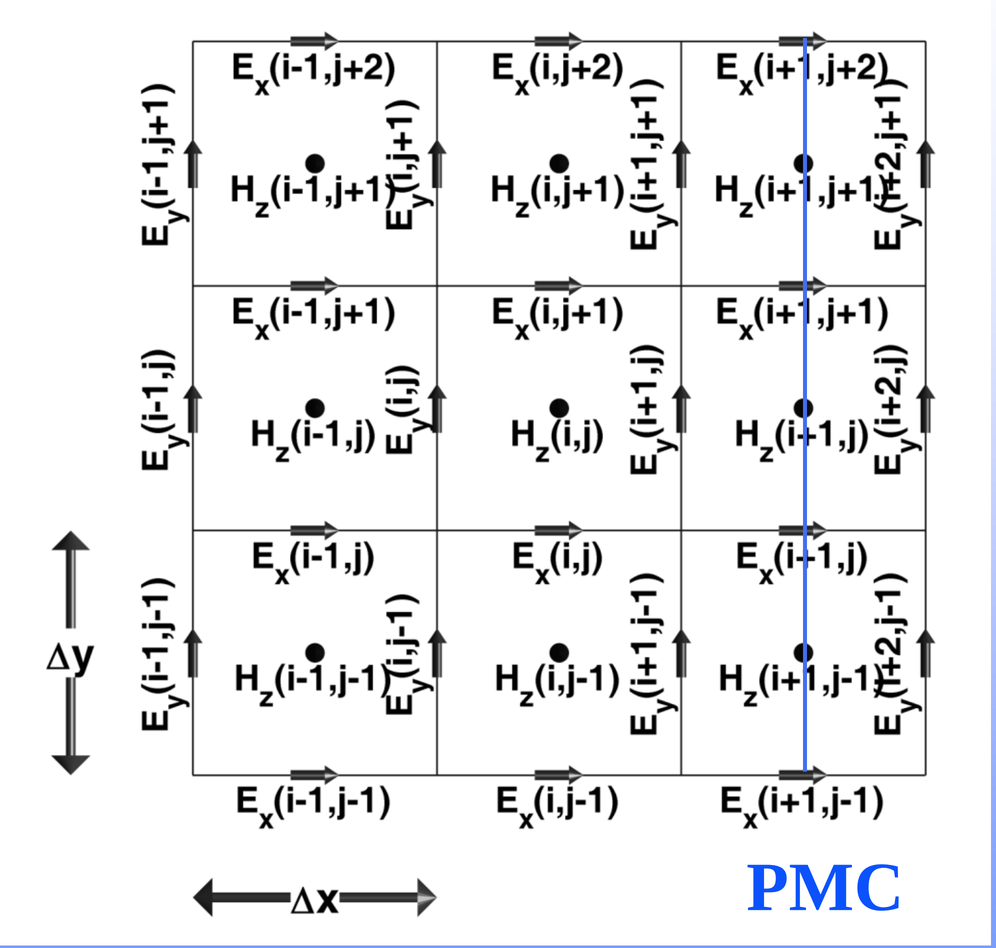
PMC

where to put the boundary magnetic walls?



PMC (Cont'd)

half a cell away from the boundary



Mur's Boundary Conditions

initial work : B. Engquist and A. Majda, "Absorbing boundary conditions for the numerical simulation of waves," Mathematics of Computation, vol. 31, 1977, pp. 629-651.

starting from the 3D wave equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}=0 \quad \Rightarrow \; Lf=0$$

$$L=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}=\partial_x^2+\partial_y^2+\partial_z^2-c^{-2}\partial_t^2$$

with respect to every dimension, the operator L is decomposed into two operators.

Mur's Boundary (Cont'd)

with respect to the x-dimension, the wave operator is decomposed to $Lf=L^+ L^- f$, where

$$L^- = \partial_x - c^{-1}\partial_t\sqrt{1-S^2},\ \ L^+ = \partial_x + c^{-1}\partial_t\sqrt{1-S^2}$$

$$S^2 = \left(\frac{\partial y}{c^{-1}\partial t}\right)^2 + \left(\frac{\partial z}{c^{-1}\partial t}\right)^2$$

the operators L^+ and L^- are pseudo-differential operators and cannot be applied directly to a function

$L^-(f)=0$ represents a wave traveling along $-x$

$L^+(f)=0$ represents a wave traveling along $+x$

Taylor expansion is used to approximate these operators

First-order Mur Boundary Condition

For a first-order approximation we use $\sqrt{1-S^2} \approx 1$

the partial derivatives with respect to *y* and *z* are assumed very small

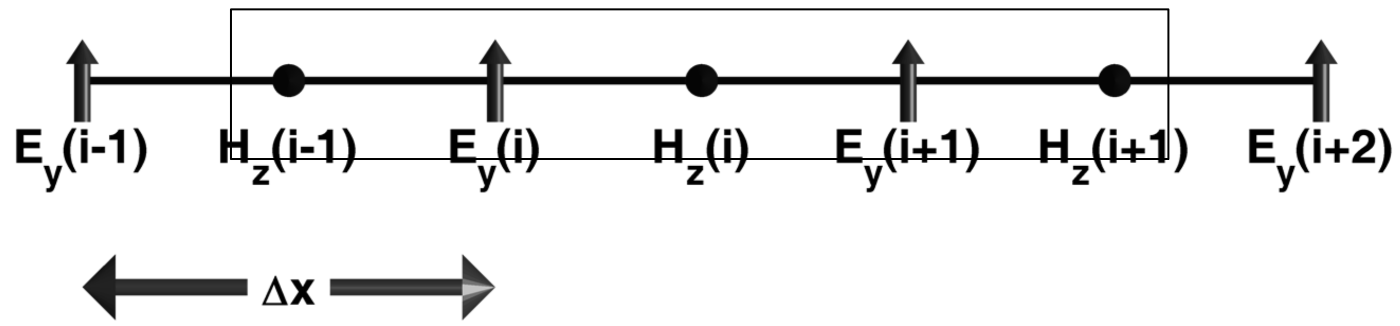
this is the case for a normally incident plane wave

$$L^- = \partial_x - c^{-1}\partial_t, \ L^+ = \partial_x + c^{-1}\partial_t$$

at x=0, we impose the condition $\dfrac{\partial f}{\partial x} - \dfrac{1}{c}\dfrac{\partial f}{\partial t} = 0$

at x=x_{max}, we impose the condition $\dfrac{\partial f}{\partial x} + \dfrac{1}{c}\dfrac{\partial f}{\partial t} = 0$

Illustration of 1st order Mur's ABC for 1D



at the left boundary, we impose the one-way condition

$$E_y^{(n+1)}(0) = E_y^{(n)}(1) + \frac{(c\Delta t - \Delta x)}{(c\Delta t + \Delta x)}\left(E_y^{(n+1)}(1) - E_y^{(n)}(0)\right)$$

the +ve x wave operator is used to derive the boundary condition at $x=x_{max}$

Second-order Mur's boundary conditions

for the second order Mur, we use the approximation

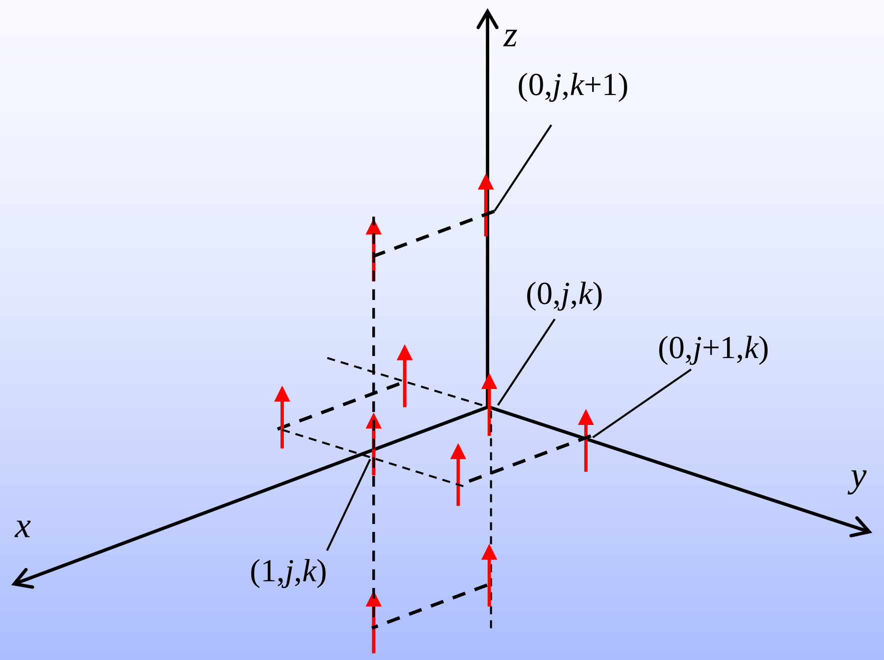
$$\sqrt{1-S^2} \approx 1-\frac{1}{2}S^2 \quad \Rightarrow \quad S^2=\left(\frac{\partial y}{c^{-1}\partial t}\right)^2+\left(\frac{\partial z}{c^{-1}\partial t}\right)^2$$

$$L^- = \partial_x - c^{-1}\partial_t\left(1-\frac{1}{2}\left(\left(\frac{\partial y}{c^{-1}\partial t}\right)^2+\left(\frac{\partial z}{c^{-1}\partial t}\right)^2\right)\right)$$

$$L^- = \partial_x - c^{-1}\partial_t + \frac{1}{2}\frac{c}{\partial_t}\left(\partial_{yy}^2 f+\partial_{zz}^2 f\right)$$

$$L^- f=\partial_{xt}^2 f - c^{-1}\partial_{tt}^2 f+0.5c\left(\partial_{yy}^2 f+\partial_{zz}^2 f\right)=0$$

Second-order Mur (Cont'd)



$$\partial_{xt}^2 f\Big|_{1/2,j,k}^{n} = \frac{1}{2\Delta t}\left(\frac{f_{1,j,k}^{n+1} - f_{0,j,k}^{n+1}}{\Delta x} - \frac{f_{1,j,k}^{n} - f_{0,j,k}^{n}}{\Delta x}\right)$$

$$\partial_{t}^2 f\Big|_{1/2,j,k}^{n} = \frac{1}{2}\left(\frac{f_{0,j,k}^{n+1} - 2f_{0,j,k}^{n} + f_{0,j,k}^{n\text{-}1}}{\Delta t^2} + \frac{f_{1,j,k}^{n+1} - 2f_{1,j,k}^{n} + f_{1,j,k}^{n\text{-}1}}{\Delta t^2}\right)$$

Second-Order Mur (Cont'd)

$$f_{0,j,k}^{n+1} = -f_{0,j,k}^{n-1} + k_1(f_{1,j,k}^{n+1} + f_{0,j,k}^{n-1}) + k_2(f_{1,j,k}^{n} + f_{0,j,k}^{n}) +$$

$$k_{3y}(f_{0,j-1,k}^{n} - 2f_{0,j,k}^{n} + f_{0,j+1,k}^{n} + f_{1,j-1,k}^{n} - 2f_{1,j,k}^{n} + f_{1,j+1,k}^{n}) +$$

$$k_{3z}(f_{0,j,k-1}^{n} - 2f_{0,j,k}^{n} + f_{0,j,k+1}^{n} + f_{1,j,k-1}^{n} - 2f_{1,j,k}^{n} + f_{1,j,k+1}^{n})$$

$$k_1 = \frac{c\Delta t - \Delta x}{c\Delta t + \Delta x} \qquad k_2 = \frac{2\Delta x}{c\Delta t + \Delta x} \qquad k_{3y} = \frac{(c\Delta t)^2 \Delta x}{2\Delta y^2 (x\Delta t + \Delta x)}$$

y and *z* derivatives can be ignored to yield a simpler expression

References

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[7] K. Kunz and R. Luebbers, *Finite-Difference Time-Domain Method for Electromagnetics*, CRC Press, 1993

[8] K.S. Yee, “Numerical solution of initial boundary value problems involving Maxwell’s equations in isotropic media,” *IEEE Trans. Antennas Propagat.*, vol. AP-14, No. 3, pp. 302-307, May 1966