EE757
Numerical Techniques in Electromagnetics
Lecture 10

Applications of MoM

- Example on static problems
- Example on 2D scattering problems
- Wire Antennas and scatterers

References

R.F. Harrington, “Field Computation by Moment Methods”

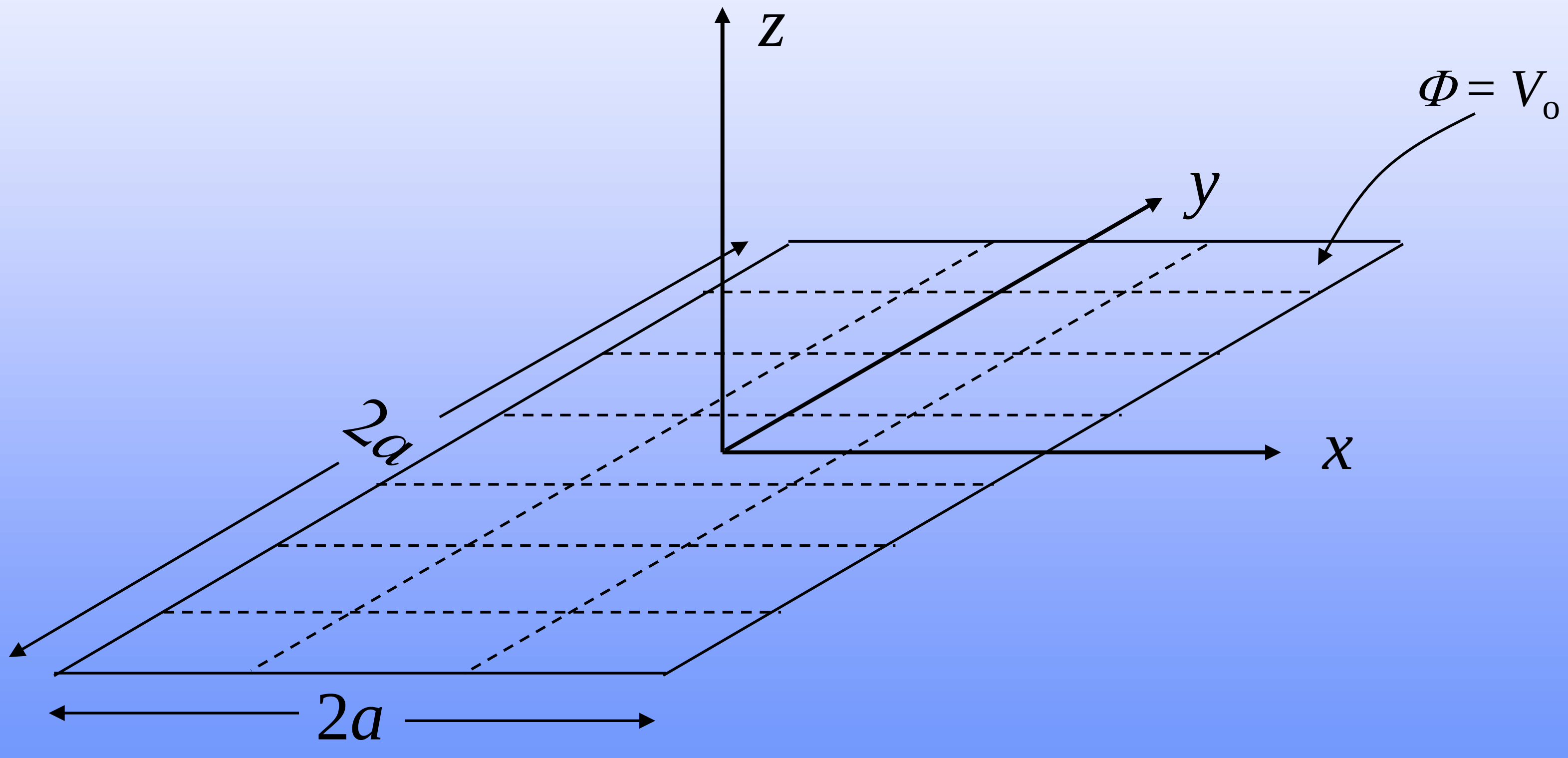
C.A. Balanis, “Advanced Engineering Electroamgnetics”

M. Sadiku,”Numerical Techniques in Electromagnetics”

S.M. Rao et al., “Electromagnetic scattering by surfaces of arbitrary shape”

A Charged Conducting Plate

- Find the charge distribution and capacitance of a metalic plate of dimensions $2a\times 2a$ whose potential is $\Phi = V_o$



A Charged Conducting Plate (Cont'd)

- The potential and charge satisfy for the unbounded medium

$$\nabla^2 \Phi = -\frac{q_{ev}}{\varepsilon}$$

- The well-known solution for this problem is

$$\Phi(\boldsymbol{r}) = \iiint_{V'} G(\boldsymbol{r},\boldsymbol{r}')\, q_{ev}(\boldsymbol{r}') dx'dy'dz'$$

$$\Downarrow$$

$$\Phi(\boldsymbol{r}) = \iiint_{V'} \frac{q_{ev}(\boldsymbol{r}')}{4\pi\varepsilon R} dx'dy'dz', \ \ R = \left|\boldsymbol{r} - \boldsymbol{r}'\right|$$

- As the plate is assumed to be in the *xy* plane we may also write

$$\Phi(x,y,z) = \int_{-a}^{a}\int_{-a}^{a} \frac{q_{es}(\boldsymbol{r}')}{4\pi\varepsilon R} dx'dy', \ \ R = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$$

A Charged Conducting Plate (Cont'd)

- We divide the conducting plate into N square subsections and define the subsectional basis function

$$f_n = \begin{cases} 1 & \text{on } \Delta S_n, \text{ the } n\text{th subsection} \\ 0, & \text{otherwise} \end{cases}$$

- We then expand the unknown surface charge density in terms of the subsectional basis functions

$$V_o = L(q_{es}) = \int_{-a}^{a}\int_{-a}^{a} \frac{q_{es}}{4\pi\varepsilon R}, \quad R = \sqrt{(x-x')^2 + (y-y')^2}$$

$$\Downarrow$$

$$V_o = \int_{-a}^{a}\int_{-a}^{a} \frac{\sum_n \alpha_n f_n}{4\pi\varepsilon R} dx'dy' = \sum_n \alpha_n \int_{-a}^{a}\int_{-a}^{a} \frac{f_n}{4\pi\varepsilon R} dx'dy'$$

A Charged Conducting Plate (Cont'd)

- But as the *nth* basis function is nonzero only over the *n*th subsection we may write

$$V_o = \sum_n \alpha_n \iint_{\Delta S_n} \frac{1}{4\pi\varepsilon R} dx'dy' \quad \text{(one equation in } N \text{ unknowns)}$$

- We utilize point matching by enforcing the above equation at the centers of each subsection

$$V_o = \sum_n \alpha_n \iint_{\Delta S_n} \frac{1}{4\pi\varepsilon R_m} dx'dy', \; R_m = \sqrt{(x_m - x')^2 + (y_m - x')^2}$$

$$m = 1, 2, \cdots, N$$

- Alternatively, $V_o = \sum_n l_{mn}\alpha_n, \; m = 1, 2, \cdots, N$

$$l_{mn} = \iint_{\Delta S_n} \frac{1}{4\pi\varepsilon R_m} dx'dy'$$

A Charged Conducting Plate (Cont'd)

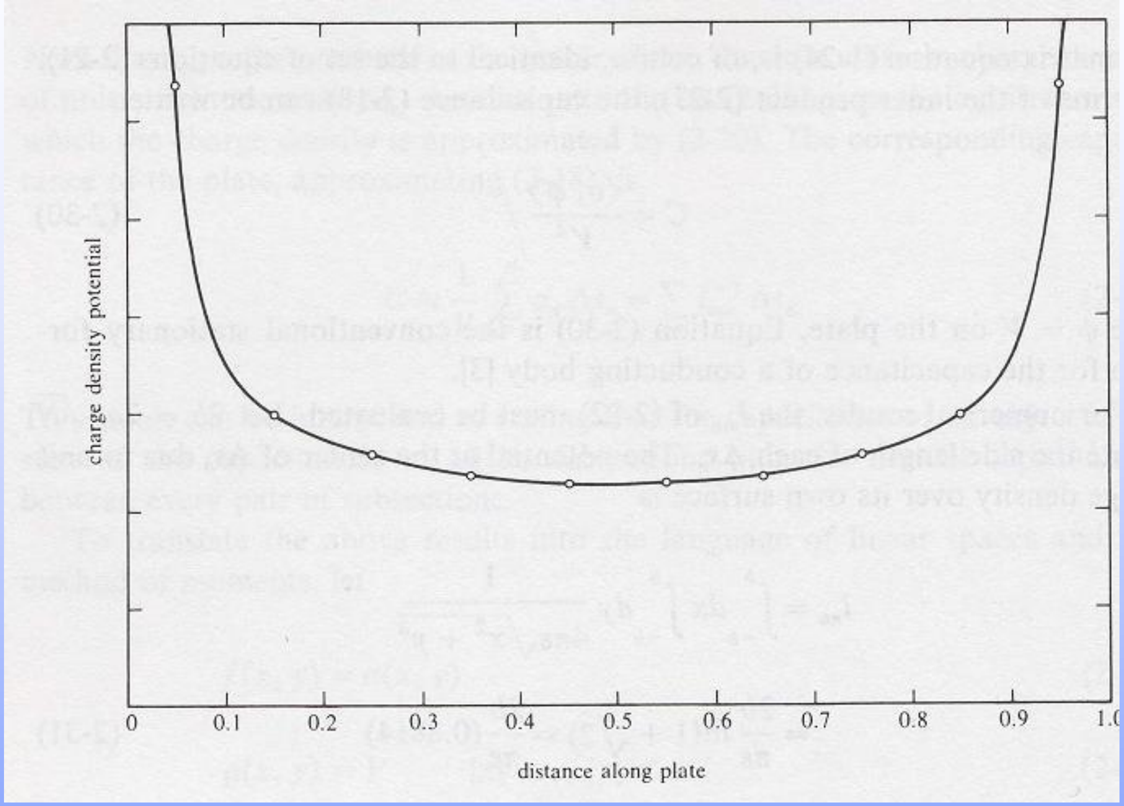
- It follows that the coefficients α_n are obtained by solving

$$\begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1N} \\ l_{21} & l_{22} & \cdots & l_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ l_{N1} & l_{N2} & & l_{NN} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} V_o \\ V_o \\ \vdots \\ V_o \end{bmatrix}$$

- Postprocessing: The capacitance of the conducting plate is approximated by

$$C = \frac{q_t}{V_o} = \frac{\sum_{n=1}^{N} \alpha_n \Delta S_n}{V_o}$$

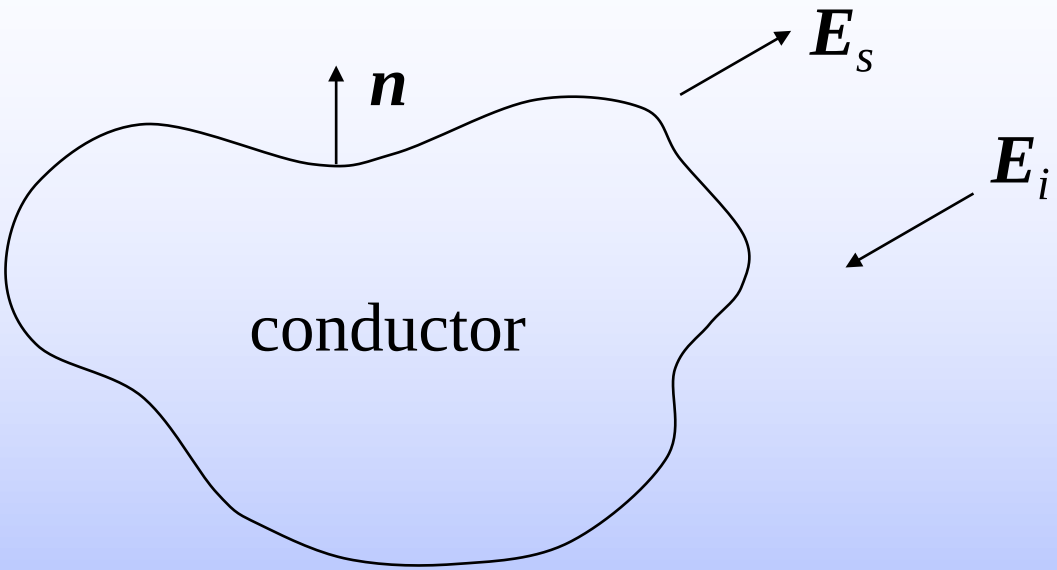
A Charged Conducting Plate (Cont'd)



Harrington, Field Computation by Moment Methods

The charge distribution along the width of the plate

Scattering Problems

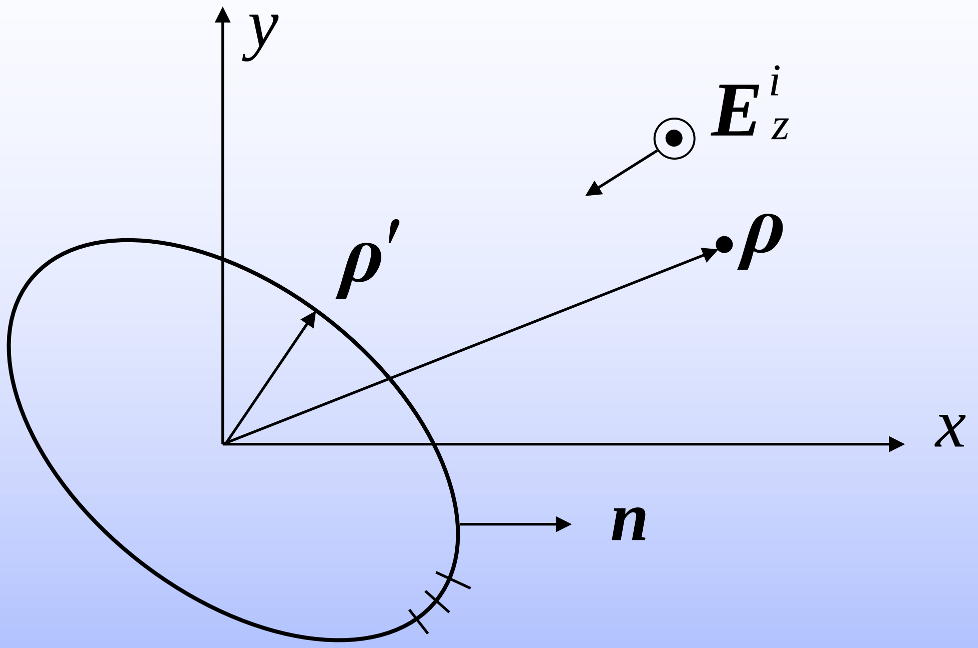


- An incident wave generates surface currents that in turn generate a scattered field such that

  $\boldsymbol{n}\times(\boldsymbol{E}_i+\boldsymbol{E}_s)=\boldsymbol{0}$ (zero total tangential electric field)

- In a scattering problem it is required to determine the surface currents. $\boldsymbol{E}_s$ is obtained as a byproduct

Scattering by a Conducting Cylinder of a TM Wave



- Incident field has only z direction $\boldsymbol{E} = E_z^i \boldsymbol{a}_z$
- Fields are dependent on x and y directions. It follows that we can solve this problem as a 2D problem

Scattering by a Conducting Cylinder (Cont'd)

- Starting with Maxwell's equations

$$(\nabla \times \boldsymbol{E}) = -j\omega\mu\boldsymbol{H}, \quad (\nabla \times \boldsymbol{H}) = \boldsymbol{J} + j\omega\varepsilon\boldsymbol{E}$$

- For the case $\boldsymbol{J}=\boldsymbol{J}_z$ we have

$$\nabla^2 E_z + k^2 E_z = j\omega\mu\, J_z$$ (We consider only the *z* component)

- The corresponding Green's function is obtained by setting

$J_z = \delta(x - x')\delta(y - y')$ to obtain

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}') = \frac{-k\eta}{4} H_o^2(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$$

- The scattered electric field is thus given by

$$E_z^s(\boldsymbol{\rho}) = -\frac{k\eta}{4} \int_{C'} J_z(\boldsymbol{\rho}') H_o^2(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|) dC'$$

Scattering by a Conducting Cylinder (Cont'd)

- For the problem at hand we must have $E_z^i = -E_z^s$ for all points on the surface of the cylinder

- It follows that we have

$$E_z^i(\boldsymbol{\rho}) = \frac{k\eta}{4} \int_{C'} J_z(\boldsymbol{\rho}') H_0^2(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|) dC', \ \forall \boldsymbol{\rho} \in C'$$

The only unknown in this equation is J_z

- We expand J_z in terms of the subsectional basis functions

$$f_n = \begin{cases} 1 & \text{on } \Delta C_n, \text{ the } n\text{th subsection} \\ 0, & \text{otherwise} \end{cases} \Rightarrow J_z = \sum_{n=1}^{N} \alpha_n f_n$$

Scattering by a Conducting Cylinder (Cont'd)

- It follows that we have

$$E_z^i(\boldsymbol{\rho}) = \frac{k\eta}{4}\sum_{n=1}^{N}\alpha_n \int_{C'} f_n\, H_o^2(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)dC',\ \forall \boldsymbol{\rho}\in C'$$

$$\Downarrow$$

$$E_z^i(\boldsymbol{\rho}) = \frac{k\eta}{4}\sum_{n=1}^{N}\alpha_n \int_{\Delta C_n} H_o^2(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)dC',\ \forall \boldsymbol{\rho}\in C'$$

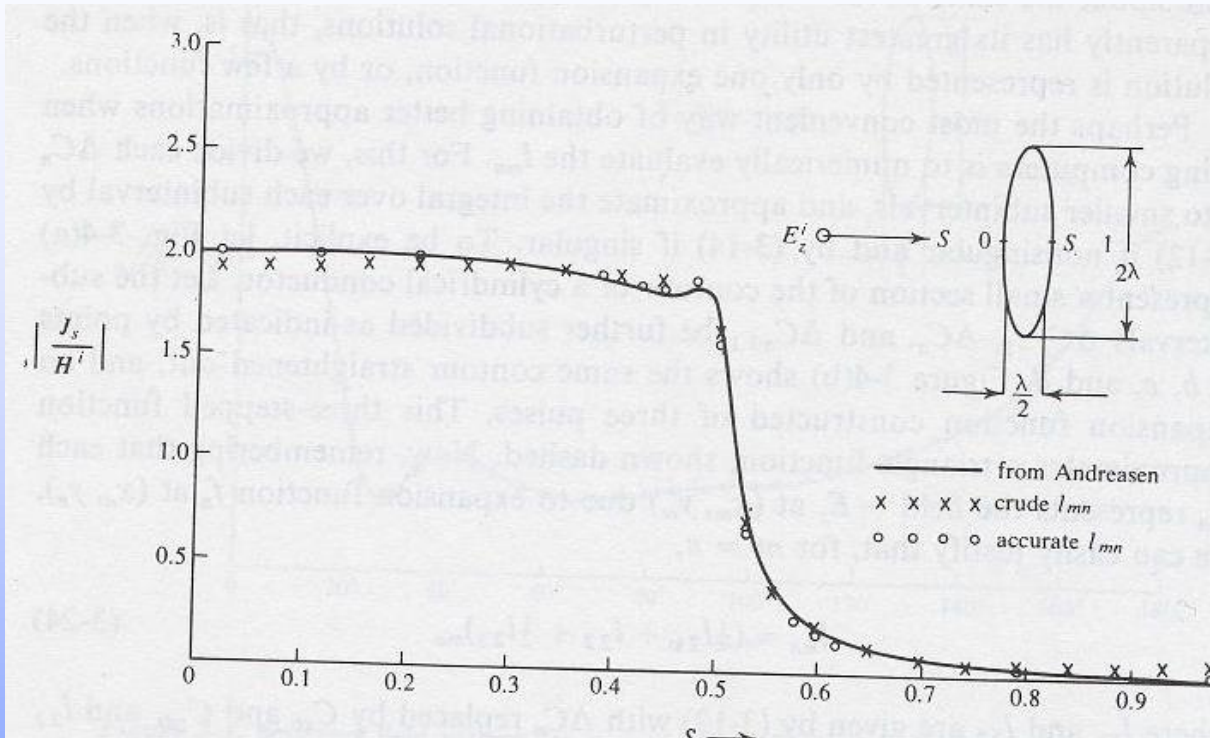
(one equation in N unknowns)

- We utilize point matching to enforce the above equation at the centers of the subsections $\boldsymbol{\rho}_m = (x_m, y_m), m = 1, 2, \cdots, N$

$$E_z^i(\boldsymbol{\rho}_m) = \frac{k\eta}{4}\sum_{n=1}^{N}\alpha_n \int_{\Delta C_n} H_o^2(k|\boldsymbol{\rho}_m-\boldsymbol{\rho}'|)dC',\ m = 1, 2, \cdots, N$$

(N equation in N unknowns)

Scattering by a Conducting Cylinder (Cont'd)

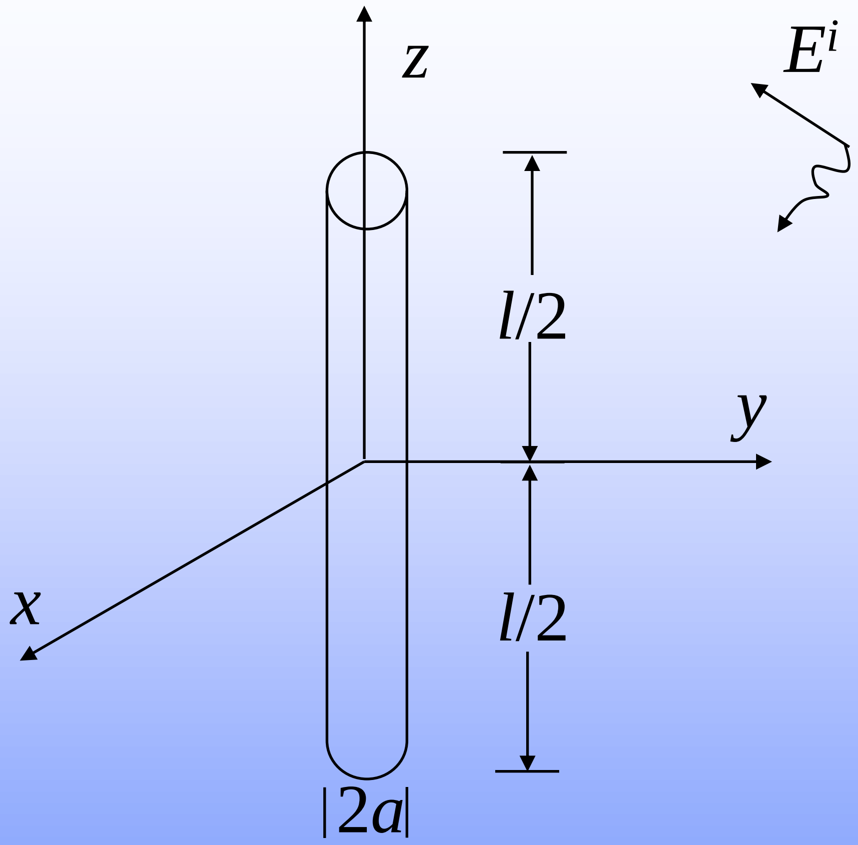


Harrington, Field Computation by Moment Methods

For a uniform plane wave incident at an angle ϕ_i we have

$$E_z^i = e^{jk(x\cos\phi_i + y\sin\phi_i)} = e^{j\boldsymbol{k}.\boldsymbol{r}}$$

Pocklington's Integral Equation



- The target is to determine the current distribution and consequently the scattered field due to an incident field for a finite-diameter wire

Pocklington's Integral Equation (Cont'd)

- The main relation for this scatterer is

$$E_z^i(\rho = a) = -E_z^s(\rho = a)$$

- The equations governing the scattered field are

$$\boldsymbol{E} = -j\omega\mathbf{A} - (j/\omega\mu\varepsilon)(\nabla(\nabla.\boldsymbol{A}))$$

- We need only the z component of the field

$$E_z^s(\boldsymbol{r}) = \frac{-j}{\omega\mu\varepsilon}\left(\beta^2 A_z + \frac{\partial^2 A_z}{\partial z^2}\right) \Rightarrow E_z^s(\boldsymbol{r}) = \frac{-j}{\omega\mu\varepsilon}\left(\beta^2 + \frac{\partial^2}{\partial z^2}\right)A_z$$

- The z component of the magnetic vector potential is

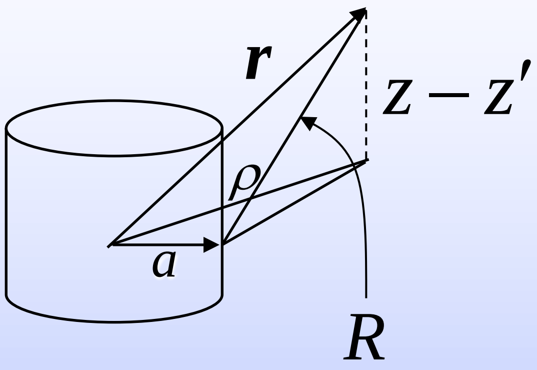
$$A_z(\boldsymbol{r}) = \frac{\mu}{4\pi}\iint_S J_z(\boldsymbol{r}')\frac{e^{-j\beta R}}{R}ds' = \frac{\mu}{4\pi}\int_{-l/2}^{l/2}\int_0^{2\pi} J_z(z',\phi')\frac{e^{-j\beta R}}{R}ad\phi'dz'$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'|$$

Pocklington's Integral Equation (Cont'd)

- If the wire is thin, J_z is not a function of ϕ

$$A_z(\mathbf{r}) = \mu \int_{-l/2}^{l/2} \underbrace{2\pi a \;\; J_z(z')}_{I_z(z')} \underbrace{\left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-j\beta R}}{4\pi R} d\phi' \right]}_{G(\mathbf{r}, z')} dz'$$



- The distance R in cylindrical coordinate is

$$R = \sqrt{\rho^2 + a^2 - 2\rho a cos(\phi - \phi') + (z - z')^2}$$

- For observation points on the wire surface we have

$$R = \sqrt{2a^2 - 2a^2 cos(\phi - \phi') + (z - z')^2}$$

$$\Downarrow$$

$$R = \sqrt{4a^2 sin^2(\frac{\phi - \phi'}{2}) + (z - z')^2}$$

Pocklington's Integral Equation (Cont'd)

- But as A_z has a ϕ symmetry, we may write

$$A_z(\rho{=}a,z,\phi) = A_z(\rho{=}a,z,0)$$

$$\Downarrow$$

$$A_z(a,z) = \mu \int_{-l/2}^{l/2} I_z(z')G(z,z')dz', \quad R = \sqrt{4a^2 \sin^2(\frac{\phi'}{2}) + (z-z')^2}$$

- The scattered field at the wire surface is thus given by

$$E_z^s(a,z) = \frac{-j}{\omega\varepsilon}(\beta^2 + \frac{\partial^2}{\partial z^2}) \int_{-l/2}^{l/2} I_z(z')G(z,z')dz'$$

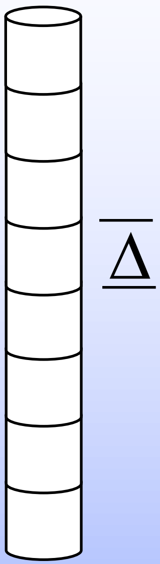
- But as $E_z^i(a,z) = -E_z^s(a,z)$, we may write

$$-j\omega\varepsilon E_z^i(a,z) = \int_{-l/2}^{l/2} I_z(z')(\beta^2 + \frac{\partial^2}{\partial z^2})G(z,z')dz'$$

Pocklington's integral equation (only I_z is not known)

Solution of Pocklington's Integral equation

- Divide the wire into N non overlapping segments
- Expand the unknown current in terms of the basis functions $I_z(z) = \sum_{n=1}^{N} I_n u_n(z)$



- For pulse functions we have

$$u_n = \begin{cases} 1, & z_{n-1/2} < z < z_{n+1/2} \\ 0, & \text{otherwise} \end{cases}$$

- For triangular functions we have

$$u_n = \begin{cases} \dfrac{\Delta - |z - z_n|}{\Delta}, & z_{n-1} < z < z_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

Solution of Pocklington's Equation (Cont'd)

- It follows that

$$-j\omega\varepsilon E_z^i(a,z)=\int_{-l/2}^{l/2}\sum_{n=1}^{N} I_n u_n(z')\left(\beta^2+\frac{\partial^2}{\partial z^2}\right)G(z,z')dz'$$

⇓

$$-j\omega\varepsilon E_z^i(a,z)=\sum_{n=1}^{N} I_n \int_{-l/2}^{l/2} u_n(z')\left(\beta^2+\frac{\partial^2}{\partial z^2}\right)G(z,z')dz'$$

⇓ using a pulse function

$$-j\omega\varepsilon E_z^i(a,z)=\sum_{n=1}^{N} I_n \int_{l_n}\left(\beta^2+\frac{\partial^2}{\partial z^2}\right)G(z,z')dz'$$

⇓

$$E_z^i(z)=\sum_{n=1}^{N} I_n G_n(z)$$

One equation in N unknowns

Solution of Pocklington's Equation (Cont'd)

- Enforcing this equation at the center of each segment, we get N equations in N unknowns

$$E_z^i(z_m) = \sum_{n=1}^{N} I_n G_n(z_m), \; m = 1, 2, \cdots, N$$

$$\begin{bmatrix} G_1(z_1) & G_2(z_1) & \cdots & G_N(z_1) \\ G_1(z_2) & G_2(z_2) & \cdots & G_N(z_2) \\ \vdots & \vdots & \vdots & \vdots \\ G_1(z_N) & G_2(z_N) & & G_N(z_N) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} E_z^i(z_1) \\ E_z^i(z_2) \\ \vdots \\ E_z^i(z_N) \end{bmatrix}$$