EE757
Numerical Techniques in Electromagnetics
Lecture 12

1D FEM

- We consider a 1D differential equation of the form

$$-\frac{d}{dx}\left(\alpha\frac{d\varphi}{dx}\right)+\beta\varphi=f,\ x\in(0,L)$$ subject to the

boundary conditions $\varphi(0)=p,\ \left[\alpha\frac{d\varphi}{dx}+\gamma\varphi\right]_{x=L}=q$

- α and β are functions associated with the physical parameters and f is the excitation

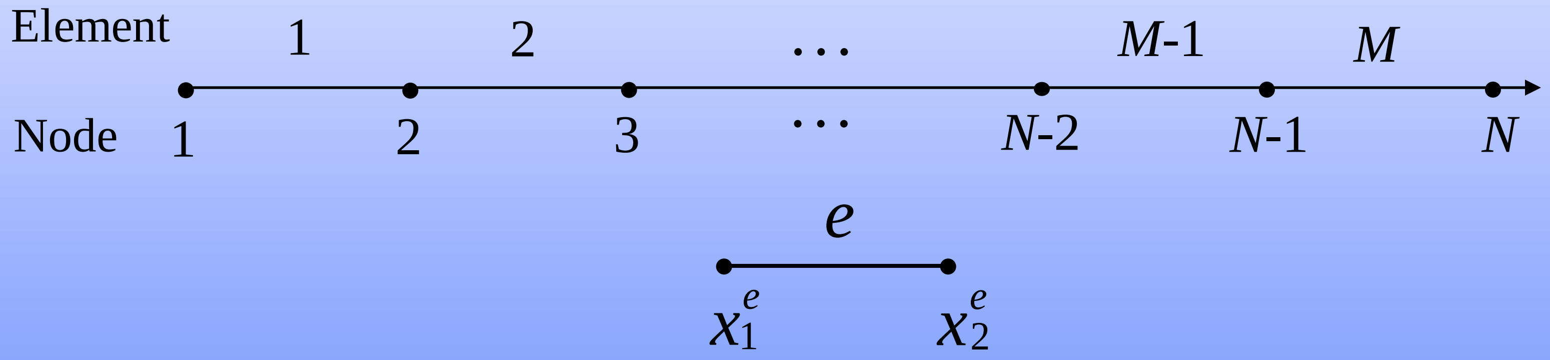
- Notice that the boundary conditions may be a Dirichlet, Neuman or mixed Dirichlet and Neuman.

1D FEM (Cont'd)

- The functional associated with this problem is

$$F(\varphi) = 0.5\int_0^L \left[\alpha \left(\frac{d\varphi}{dx} \right)^2 + \beta \varphi^2 \right] dx - \int_0^L f\varphi \, dx + \left[\frac{\gamma}{2}\varphi^2 - q\varphi \right]_{x=L}$$

(Prove it)!



- We divide our computational domain into M elements with a total number of N nodes

1D FEM (Cont'd)

e \ j	1	2
1	1	2
2	2	3
3	3	4
:	:	:
M	M	M+1

- We utilize an index table to determine the global index of each local node

- For this simple 1D case, we have N=M+1 and

$$x_1^e = x_e, \quad x_2^e = x_{e+1}$$

1D FEM (Cont'd)

- We approximate the unknown function φ by a linear approximation over each element, i.e.,

$$\varphi_e(x) = a^e + b^e x, \quad x \in \Omega_e$$

- It follows that we have

$$\varphi_1^e = a^e + b^e x_1^e, \qquad \varphi_2^e = a^e + b^e x_2^e$$

⇓ Express a^e and b^e in terms of nodes values

$$\varphi^e(x) = \sum_{j=1}^{2} N_j^e(x)\,\varphi_j^e$$

- $N_j^e(x)$ is the jth interpolation function of the eth element where $N_1^e(x) = (x_2^e - x)/l^e$ and $N_2^e(x) = (x - x_1^e)/l^e$ where l^e is the length of the eth element, i.e., $l^e = x_2^e - x_1^e$

The Homogenous Neuman BC Case

- For this case ($\gamma = q = 0$), the functional is given by

$$F(\varphi) = 0.5\int_0^L \left[\alpha \left(\frac{d\varphi}{dx} \right)^2 + \beta \varphi^2 \right] dx - \int_0^L f\varphi \, dx$$

⇓ Use elemental expansion

$$F(\varphi) = 0.5 \sum_{e=1}^{M} \int_{x_1^e}^{x_2^e} \left[\alpha \left(\frac{d\varphi^e}{dx} \right)^2 + \beta (\varphi^e)^2 \right] dx - \sum_{e=1}^{M} \int_{x_1^e}^{x_2^e} f \varphi^e \, dx$$

⇓ F can be written as a sum of subfunctionals

$$F(\varphi) = \sum_{e=1}^{M} F^e(\varphi^e)$$

The Homogenous Neuman BC Case (Cont'd)

- The eth subfunctional is thus given by

$$F^e(\varphi^e) = 0.5\int_{x_1^e}^{x_2^e}\left[\alpha\left(\frac{d\varphi^e}{dx}\right)^2 + \beta(\varphi^e)^2\right]dx - \int_{x_1^e}^{x_2^e} f\,\varphi^e\,dx$$

- It follows that we have

$$\frac{\partial F(\varphi)}{\partial \varphi_i} = \sum_{e=1}^{M}\frac{\partial F^e(\varphi^e)}{\partial \varphi_i},\ i = 1, 2, \cdots, N$$

- Notice the coefficient of any node value may be obtained through summing the associated coefficients obtained by differentiating each subfunctional

- Substituting in the eth subfunctional with the expansion $\varphi^e(x) = \sum_{j=1}^{2} N_j^e(x)\,\varphi_j^e$ we get

The Homogenous Neuman BC Case (Cont'd)

- It follows that we have

$$F^e(\varphi^e) = 0.5\int_{x_1^e}^{x_2^e}\left[\alpha\sum_{i=1}^{2}\sum_{j=1}^{2}\varphi_i^e \frac{d\,N_i^e}{dx}\frac{d\,N_j^e}{dx}\varphi_j^e + \beta\,\varphi_i^e N_i^e N_j^e \varphi_j^e\right]dx$$

$$-\int_{x_1^e}^{x_2^e} f \sum_{i=1}^{2}\varphi_i^e N_i^e\,dx$$

⇓ Differentiating w.r.t. φ_i^e

$$\frac{d\,F^e}{d\,\varphi_i^e} = \sum_{j=1}^{2}\varphi_j^e\left(\int_{x_1^e}^{x_2^e}\alpha\frac{d\,N_i^e}{dx}\frac{d\,N_j^e}{dx} + \beta\,N_i^e N_j^e\,dx\right) - \int_{x_1^e}^{x_2^e} f\,N_i^e\,dx$$

- Or in matrix form $\left\{\frac{\partial F^e}{\partial \boldsymbol{\varphi}^e}\right\}_{2\times1} = \left[\boldsymbol{K}^e\right]_{2\times2}\left[\boldsymbol{\varphi}^e\right]_{2\times1} - \left[\boldsymbol{b}^e\right]_{2\times1}$

The homogenous Neuman BC case (Cont'd)

- The coefficients of the matrix $\boldsymbol{K}^e$ and the vector $\boldsymbol{b}^e$ are

$$K_{ij}^e = \int_{x_1^e}^{x_2^e} \left(\alpha \frac{d\, N_i^e}{dx} \frac{d\, N_j^e}{dx} + \beta N_i^e N_j^e \right) dx$$

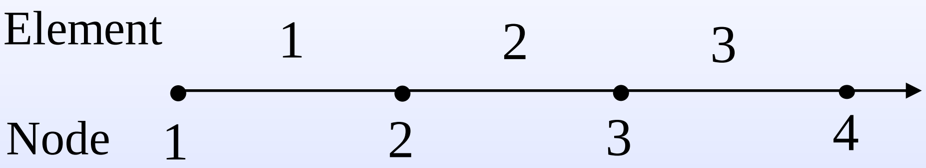
$$b_i^e = \int_{x_1^e}^{x_2^e} f\, N_i^e\, dx$$

- The process of assembly involves storing the local elemental components into their proper location in the system of equations

$$\frac{\partial F}{\partial \boldsymbol{\varphi}} = \mathbf{0} \quad \Longrightarrow \quad [\boldsymbol{K}]_{N\times N} [\boldsymbol{\varphi}]_{N\times 1} = [\boldsymbol{b}]_{N\times 1}$$

An Assembly Example

- assuming that we have only 3 elements (4 unknowns)



- Initialization: $\boldsymbol{K}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \boldsymbol{b}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

- 1st element: $\boldsymbol{K}^{(1)}=\begin{bmatrix}K_{11}^{(1)}&K_{12}^{(1)}\\K_{21}^{(1)}&K_{22}^{(1)}\end{bmatrix}$, and $\boldsymbol{b}^{(1)}=\begin{bmatrix}b_1^{(1)}\\b_2^{(1)}\end{bmatrix}$

- Update the global system to get

An Assembly Example (Cont'd)

$$\boldsymbol{K}=\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{b}=\begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ 0 \\ 0 \end{bmatrix}$$

- 2nd element: $\boldsymbol{K}^{(2)}=\begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} \end{bmatrix}$, and $\boldsymbol{b}^{(2)}=\begin{bmatrix} b_1^{(2)} \\ b_2^{(2)} \end{bmatrix}$
- Update the global system to get

$$\boldsymbol{K}=\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)}+K_{11}^{(2)} & K_{12}^{(2)} & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{b}=\begin{bmatrix} b_1^{(1)} \\ b_2^{(1)}+b_1^{(2)} \\ b_2^{(2)} \\ 0 \end{bmatrix}$$

An Assembly Example (Cont'd)

- By assembling the 3rd element we finally have the system of equations

$$\boldsymbol{K} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} \\ 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)} \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} + b_1^{(2)} \\ b_2^{(2)} + b_1^{(3)} \\ b_2^{(3)} \end{bmatrix}$$

The General Boundary Case

- For the case $q \neq 0$ or $\gamma \neq 0$, the functional is augmented by the subfunctional

$$F_b(\varphi) = \left[\frac{\gamma}{2} \varphi^2 - q\varphi \right]_{x=L} \Rightarrow F_b(\varphi) = \frac{\gamma}{2} \varphi_N^2 - q\,\varphi_N$$

$$\frac{\partial F_b}{\partial \varphi_N} = \gamma\,\varphi_N - q$$

- It follows that only K_{NN} is incremented by γ and b_N is incremented by q

Dirichlet's Boundary Conditions

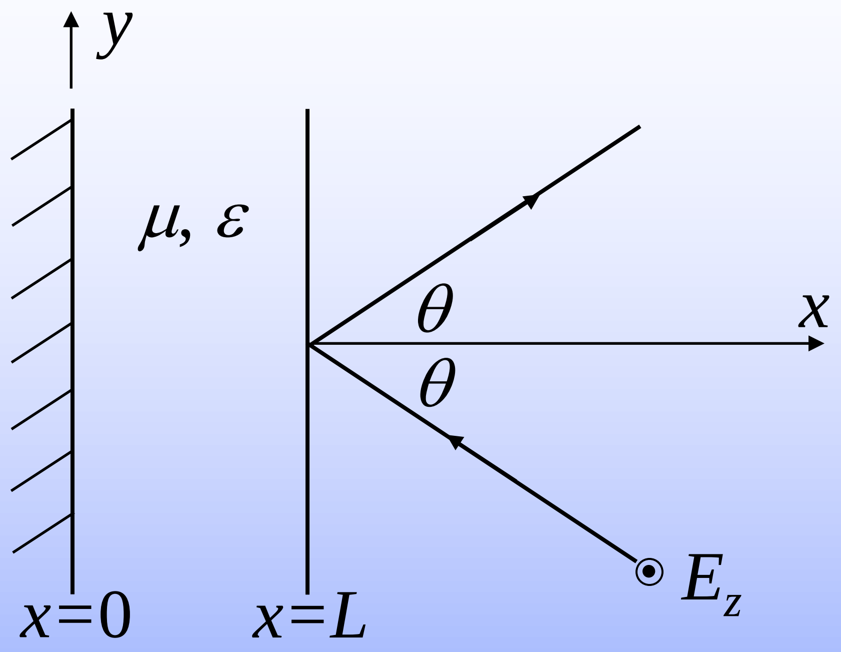
- Dirichlet's boundary conditions are imposed by eliminating the corresponding nodal values
- Example: for the case N=4, we have

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$\Downarrow$ if $\varphi_1 = p$

$$\boldsymbol{K} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & K_{22} & K_{23} & K_{24} \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} p \\ b_2 - K_{21} p \\ b_3 - K_{31} p \\ b_4 - K_{41} p \end{bmatrix} \Rightarrow \begin{bmatrix} K_{22} & K_{23} & K_{24} \\ K_{32} & K_{33} & K_{34} \\ K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} b_2 - K_{21} p \\ b_3 - K_{31} p \\ b_4 - K_{41} p \end{bmatrix}$$

Example



- Determine the power reflected by this inhomogeneous metal-backed dielectric slab for a uniform incident plane wave with a *z* polarized electric field. Both μ_r and ε_r may vary with *x*.

Example (Cont'd)

- The expression for the incident electric field is

$$E_z^{inc}(x,y)=E_{\text{o}}\exp(-j\boldsymbol{k}.\boldsymbol{r}),\quad \boldsymbol{k}=-k_o\cos\theta\,\boldsymbol{a}_x+k_o\sin\theta\,\boldsymbol{a}_y$$

$$\Downarrow$$

$$E_z^{inc}(x,y)=E_{\text{o}}\exp(j\,k_{\text{o}}\,x\cos\theta-j k_{\text{o}}\,y\sin\theta)$$

- Notice that all field components must have a variation of $\exp(-j\,k_o\,y\sin\theta)$ to satisfy field continuity
- The wave equation for this problem is

$$\nabla\times\left(\frac{1}{\mu}\nabla\times\boldsymbol{E}\right)-\omega^2\varepsilon\boldsymbol{E}=-j\omega\boldsymbol{J}$$

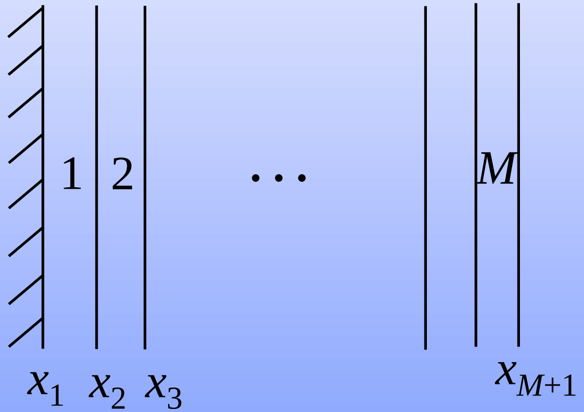
- Taking into account that $\boldsymbol{E}=E_z\boldsymbol{a}_z$, $\partial/\partial z=0$ and $\boldsymbol{J}=\boldsymbol{0}$, we get

$$\left[\frac{\partial}{\partial x}\left(\frac{1}{\mu_r}\frac{\partial}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\mu_r}\frac{\partial}{\partial y}\right)+k_{\text{o}}^2\varepsilon_r\right]E_z=0$$

Example (Cont'd)

- Substituting for $\frac{\partial}{\partial y}\left(\frac{1}{\mu_r}\right)=0,$ and $\frac{\partial E_z}{\partial y}=-j k_o \sin\theta\, E_z$ we get

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu_r}\frac{dE_z}{dx}\right)+k_o^2\left(\varepsilon_r-\frac{\sin^2\theta}{\mu_r}\right)E_z=0 \text{ with the BC } E_z(0)=0$$



- An analytical solution is obtained by dividing the slab into layers m=1, 2, …, M , where ε_r and μ_r are assumed constant with values ε_{rm} and μ_{rm}, respectively.

Example (Cont'd)

- The wave equation in each layer, thus becomes

$$\frac{1}{\mu_{rm}}\frac{d^2 E_z}{d x^2}+k_o^2(\varepsilon_{rm}-\frac{\sin^2\theta}{\mu_{rm}})E_z=0 \Rightarrow \frac{d^2 E_z}{d x^2}+k_o^2(\varepsilon_{rm}\mu_{rm}-\sin^2\theta)E_z=0$$

which has the solution

$$E_{zm}=(A_m\exp(j k_{xm} x)+B_m\exp(-jk_{xm} x))\exp(-j k_o\sin\theta\, y)$$

$$k_{xm}=k_o\sqrt{\varepsilon_{rm}\mu_{rm}-\sin^2\theta}$$

- The analytical solution is obtained by enforcing continuity of the electric and magnetic field components at the layers interface to get

Example (Cont'd)

$$R_{m+1} = \frac{\eta_{m+1,m} + R_m \exp(-2jk_{xm}x_{m+1})}{1 + \eta_{m+1,m} R_m \exp(-2jk_{xm}x_{m+1})} \exp(2jk_{xm+1}x_{m+1})$$

$$R_m = \frac{B_m}{A_m}, \;\; \eta_{m+1,m} = \frac{\mu_{rm} k_{xm+1} - \mu_{rm+1} k_{xm}}{\mu_{rm} k_{xm+1} + \mu_{rm+1} k_{xm}}$$

$$R_1 = \frac{B_1}{A_1} = -1, \quad \text{(conductor)}$$ **(Prove it)!**

FEM Solution

- Our problem is given by

$$\frac{d}{dx}\left(\frac{1}{\mu_r}\frac{dE_z}{dx}\right)+k_o^2\left(\varepsilon_r-\frac{\sin^2\theta}{\mu_r}\right)E_z=0 \text{ with the BC } E_z(0)=0$$

- A boundary condition at $x=L$ to have a finite computational domain
- For $L\leq x$, we have

$$E_z(x,y)=(E_0\exp(jk_o x\cos\theta)+RE_0\exp(-jk_o x\cos\theta))\exp(-jk_o y\sin\theta)$$

⇩

$$E_z(x,y)=E_z(x)\exp(-jk_o y\sin\theta)$$

- Differentiating $E_z(x)$ relative to x we get

$$\frac{dE_z(x)}{dx}=jk_o\cos\theta(E_0\exp(jk_o x\cos\theta)-RE_0\exp(-jk_o x\cos\theta))$$

FEM Solution (Cont'd)

- Manipulating we get

$$\frac{dE_z(x)}{dx} = 2jk_o\cos\theta E_0\exp(jk_o x\cos\theta) - jk_o\cos\theta E_z(x)$$

$$\Downarrow$$

$$\left.\left(\frac{dE_z(x)}{dx} + jk_o\cos\theta E_z(x)\right)\right|_{x=L^+} = 2jk_o\cos\theta E_0\exp(jk_o L\cos\theta)$$

- Utilizing the continuity of the electric and magnetic field we may convert this boundary condition into

$$\left.\left(\frac{1}{\mu_r}\frac{dE_z(x)}{dx} + jk_o\cos\theta E_z(x)\right)\right|_{x=L^-} = 2jk_o\cos\theta E_0\exp(jk_o L\cos\theta)$$

FEM Solution (Cont'd)

- It follows that our problem is given by

$$\frac{d}{dx}\left(\frac{1}{\mu_r}\frac{dE_z}{dx}\right)+k_o^2\left(\varepsilon_r-\frac{\sin^2\theta}{\mu_r}\right)E_z=0 \quad \text{with} \quad E_z(0)=0$$

$$\left.\left(\frac{1}{\mu_r}\frac{dE_z(x)}{dx}+jk_o\cos\theta E_z(x)\right)\right|_{x=L}=2jk_o\cos\theta E_o\exp(jk_oL\cos\theta)$$

- Comparing with our 1D FEM formulation we have

$$\varphi=E_z(x),\quad \alpha=1/\mu_r,\quad \beta=-k_o^2\left(\varepsilon_r-\frac{\sin^2\theta}{\mu_r}\right)$$

$$\gamma=jk_o\cos\theta,\quad q=2jE_ok_o\cos\theta\exp(k_oL\cos\theta)$$

- Once the field is solved, the reflection coefficient is given by

$$R=\frac{E_z(x)-(E_o\exp(jk_oL\cos\theta)}{E_o\exp(-jk_oL\cos\theta)}$$