# EE757
# Numerical Techniques in Electromagnetics
# Lecture 14

# Nodal FEM vs. Vector FEM

- Nodal FEM is suitable for scalar problems or for field problems involving one field component only
- Nodal FEM in some problems can give rise to false (spurious) solutions
- Vector FEM is more suitable for field vector quantities
- The field over the 2D or 3D element is expressed as a linear combination of vector basis functions

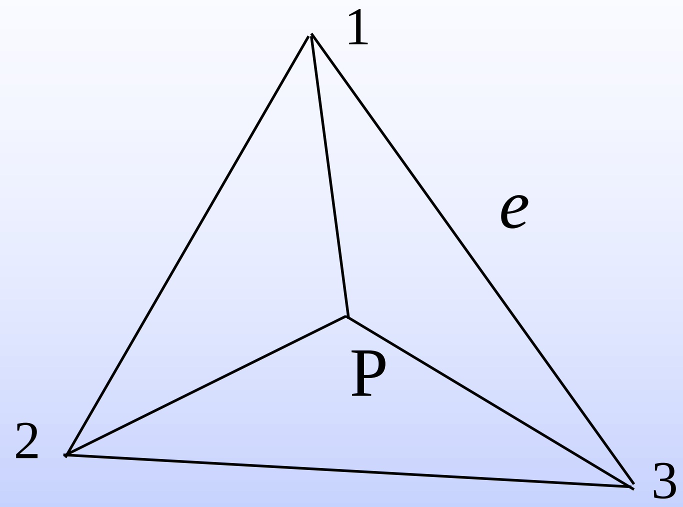
## Area Coordinates

$\Delta$=area of triangle 123

$$\Delta = \begin{vmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{vmatrix}$$

$\Delta_1$ =area of triangle P23

$$\Delta_1 = \begin{vmatrix} 1 & x & y \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{vmatrix}$$




$\Delta_2$ =area of triangle P13

$$\Delta_2 = \begin{vmatrix} 1 & x & y \\ 1 & x_1^e & y_1^e \\ 1 & x_3^e & y_3^e \end{vmatrix}$$

$\Delta_3$ =area of triangle P12

$$\Delta_3 = \begin{vmatrix} 1 & x & y \\ 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \end{vmatrix}$$

## Area Coordinates (Cont'd)

- We define the nodal basis functions

$$N_1(x,y)=\frac{\Delta_1}{\Delta},\ \ N_2(x,y)=\frac{\Delta_2}{\Delta},\ \ N_3(x,y)=\frac{\Delta_3}{\Delta}$$

- Notice that these basis functions satisfy

$$N_i(x_j^e, y_j^e)=\delta_{ij},$$

$$N_1(x,y)+N_2(x,y)+N_3(x,y)=1,\ \ x,y\in\Omega_e$$

- A possible expansion for 2D nodal FEM is

$$\varphi(x,y)=\sum_{j=1}^{3}N_j(x,y)\,\varphi_j^e$$

# Vector (Edge) FEM

- We expand the unknown vector quantity using the expansion $\boldsymbol{F} = \sum_{j=1}^{3} F_j^e \boldsymbol{W}_j^e$
- $F_j^e$ is the average value of the tangential field quantity over the *j*th edge
- $\boldsymbol{W}_j^e$ is the edge basis function related to the *j*th edge
- One of the most commonly used edge basis functions are the Whitney basis functions given by

$$\boldsymbol{W}_1^e = \ell_{12}(N_1 \nabla N_1 - N_2 \nabla N_2)$$

$$\boldsymbol{W}_2^e = \ell_{23}(N_2 \nabla N_2 - N_3 \nabla N_3)$$

$$\boldsymbol{W}_3^e = \ell_{31}(N_3 \nabla N_3 - N_1 \nabla N_1)$$

# Vector FEM (Cont'd)

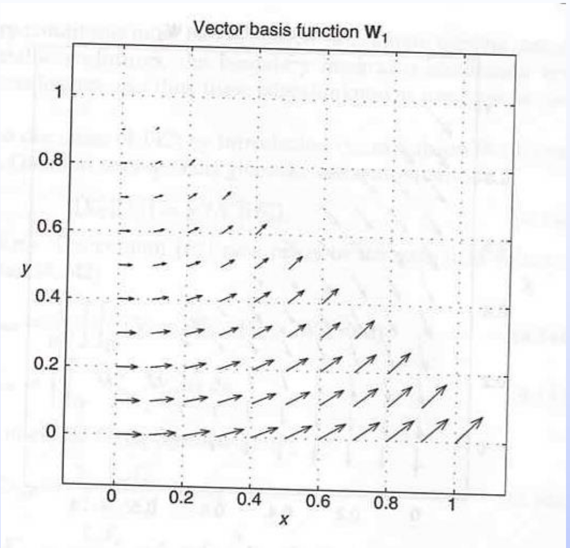
- The following properties can be proven for Whitney's basis functions:

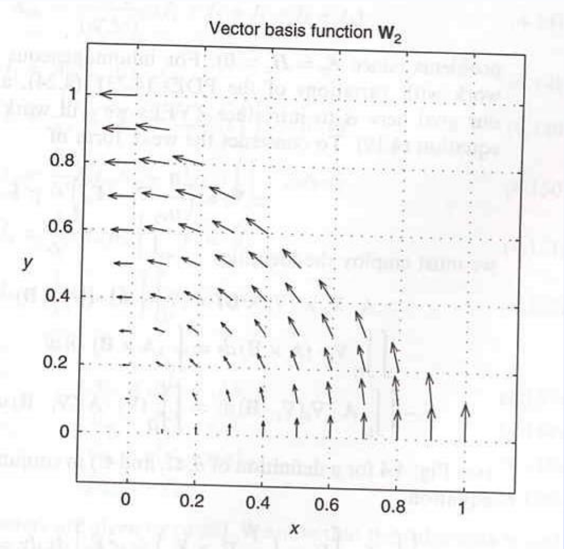
$\boldsymbol{W}_k^e \cdot \boldsymbol{e}_k = 1.0$ ⇨ unity tangential component along its edge

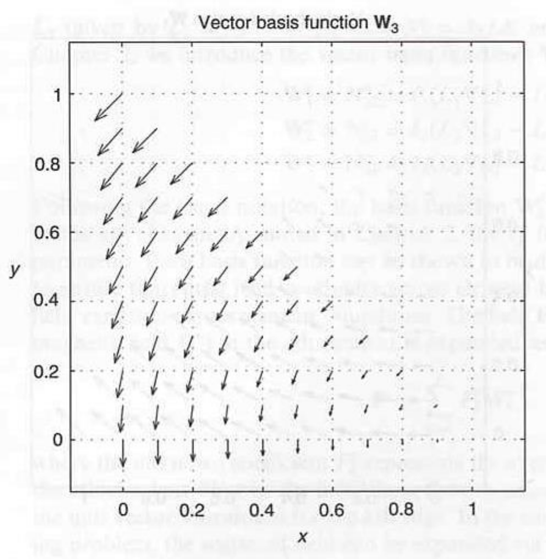
$\boldsymbol{W}_k^e \cdot \boldsymbol{e}_j = 0, \; j \neq k$ ⇨ Only normal component along other edges

$\nabla . \boldsymbol{W}_k^e = 0,$ ⇨ Divergenless over element

# Vector FEM (Cont'd)




*Volakis et al.*, Finite Element Method for Electromagnetics

# Example

- The vector wave equation with no sources is given by

$$\nabla_t \times \left( \frac{1}{\mu_r} \nabla_t \times \boldsymbol{E}_t \right) - (k_o^2 \varepsilon_r - \beta^2) \boldsymbol{E}_t = \boldsymbol{0}$$

$$\Downarrow$$

$$\nabla_t \times \left( \frac{1}{\mu_r} \nabla_t \times \boldsymbol{E}_t \right) - \gamma^2 \boldsymbol{E}_t = \boldsymbol{0}$$

- Using Galerkin's approach we get

$$\iint_{\Omega} \boldsymbol{T}_i \cdot \left[ \nabla_t \times \left( \frac{1}{\mu_r} \nabla_t \times \boldsymbol{E}_t \right) - \gamma^2 \boldsymbol{E}_t \right] dxdy = \boldsymbol{0}$$

- Using vector identities we have

# Example (Cont'd)

$$\iint_{\Omega}\left[(\nabla_t \times \boldsymbol{T}_i).\left(\frac{1}{\mu_r}\nabla_t \times \boldsymbol{E}_t\right) - \gamma^2 \boldsymbol{T}_i.\boldsymbol{E}_t\right] dxdy = F = \mathbf{0}$$

- Utilizing the vector expansion

$\boldsymbol{E}_t = \sum_{j=1}^{3} E_j^e \boldsymbol{W}_j^e$ over the *e*th element, we have the elemental subfuntional

$$F_e = \sum_{j=1}^{3} E_j^e \iint_{\Omega e}\left[(\nabla_t \times \boldsymbol{T}_i).\left(\frac{1}{\mu_r}\nabla_t \times \boldsymbol{W}_j^e\right) - \gamma^2 \boldsymbol{T}_i.\boldsymbol{W}_j^e\right] dxdy$$

- Using $\boldsymbol{T}_i = \boldsymbol{W}_i^e$, we have

$$F_e = \sum_{j=1}^{3} E_j^e \iint_{\Omega e}\left[(\nabla_t \times \boldsymbol{W}_i^e).\left(\frac{1}{\mu_r}\nabla_t \times \boldsymbol{W}_j^e\right) - \gamma^2 \boldsymbol{W}_i^e.\boldsymbol{W}_j^e\right] dxdy$$

# Example (Cont'd)

- In matrix form, we have

$$F_e = \left(\boldsymbol{K}_{\nabla}^{e} - \gamma^2 \boldsymbol{K}^{e}\right)\boldsymbol{E}^{e},$$

where

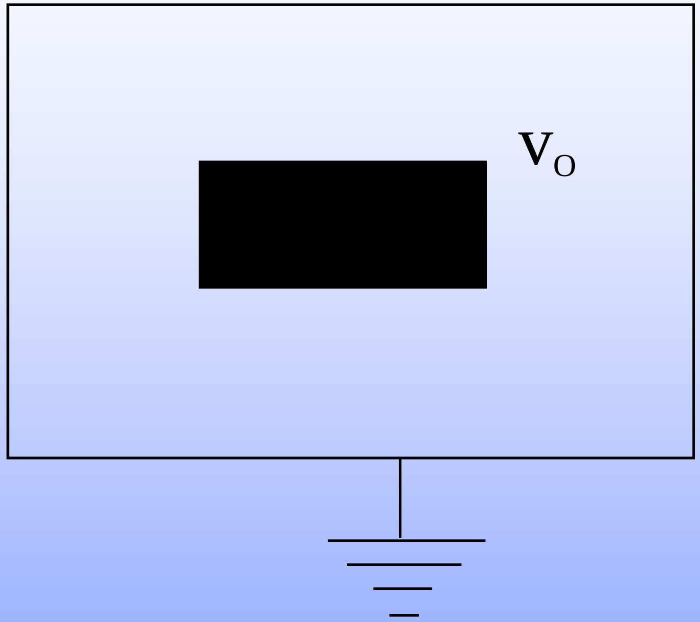
$$K_{\nabla mn}^{e} = \frac{1}{\mu_r^e} \iint_{\Omega e} \left[(\nabla_t \times \boldsymbol{W}_m^e).(\nabla_t \times \boldsymbol{W}_n^e)\right] dxdy$$

$$K_{mn}^{e} = \iint_{\Omega e} \boldsymbol{W}_m^e . \boldsymbol{W}_n^e \, dxdy$$

- The process of assembly is then utilized to solve the global eigenvalue problem

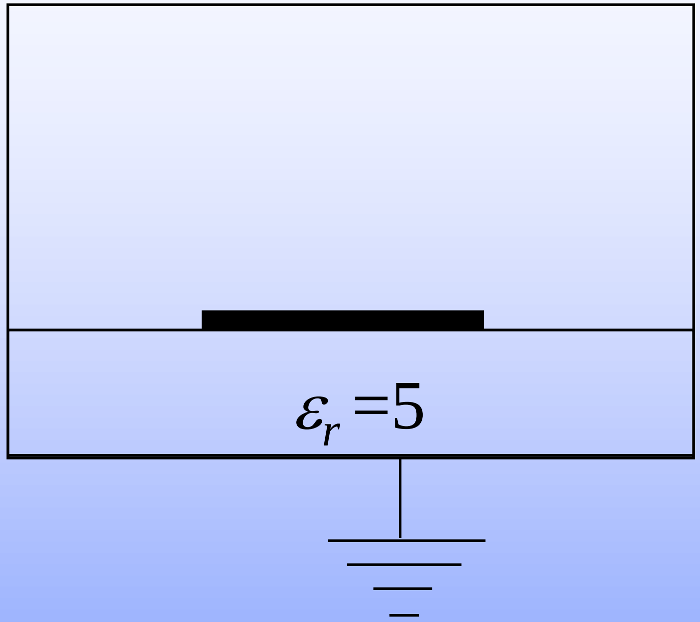
$$\left(\boldsymbol{K}_{\nabla} - \gamma^2 \boldsymbol{K}\right)\boldsymbol{E} = \boldsymbol{0}$$

# Project 3




Utilize the FEM method to draw the equipotential lines and the field for this square coaxial cable

# Project 3 (Cont'd)




Find the capacitance and the characteristic impedance for the shown shielded microstrip line using the finite element method

# Project 3 (Cont'd)

Apply a vector finite element approach to find the cut-off frequencies of a rectangular waveguide.