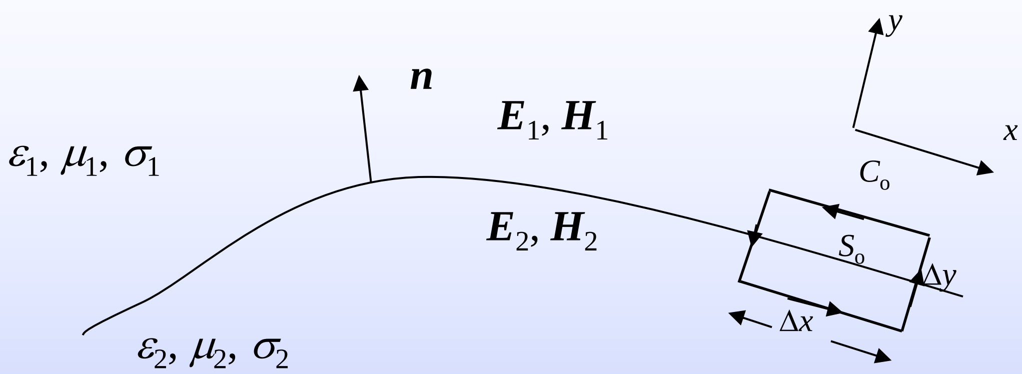
EE757
Numerical Techniques in Electromagnetics
Lecture 2

Boundary Conditions

- Maxwell's Equations are partial differential equations
- Boundary conditions are needed to obtain a unique solution
- Maxwell's differential equations do not apply on the boundaries because the fields are discontinuous
- Our target is to determine the electric and magnetic fields in a certain region of space due to excitations satisfying the problem's boundary conditions

Finite Conductivity Case



- Applying Faraday's law we get $\oint_{C_o} \boldsymbol{E}.\boldsymbol{dl} = -\dfrac{\partial}{\partial t}\iint_{S_o} \boldsymbol{B}.\boldsymbol{dS}$
- As $\Delta y \rightarrow 0$, the RHS vanishes and we get

$$\boldsymbol{E}_2.\Delta x\boldsymbol{a}_x - \boldsymbol{E}_1.\Delta x\boldsymbol{a}_x = 0 \quad \Longrightarrow \quad E_1^t = E_2^t$$

or alternatively, $\boldsymbol{n}\times(\boldsymbol{E}_1 - \boldsymbol{E}_2) = \boldsymbol{0}$

- It follows that the tangential component of the electric field is continuous (no magnetic current is assumed)

Finite Conductivity Case (Cont'd)

- Similarly, starting with the modified Ampere's law

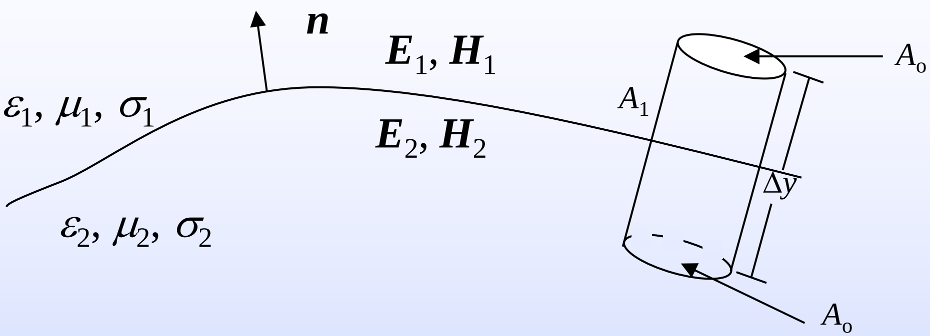
$\oint_{C_o} \boldsymbol{H}.d\boldsymbol{l} = \frac{\partial}{\partial t}\iint_{S_o} \boldsymbol{D}.d\boldsymbol{S}$ (no current $\boldsymbol{J}$ at the interface), we get

$$\boldsymbol{H}_1.\Delta x\,\boldsymbol{a}_x - \boldsymbol{H}_2.\Delta x\,\boldsymbol{a}_x = 0 \quad \Longrightarrow \quad H_1^t = H_2^t$$

or alternatively, $\boldsymbol{n}\times(\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{0}$

- It follows that the tangential component of the magnetic field intensity is continuous if there are no boundary electric currents

Finite Conductivity Case (Cont'd)



- Assuming there are no surface charges, Gauss's law gives $\oiint_S \boldsymbol{D}.d\boldsymbol{S} = Q_{ev}$ $\Longrightarrow$ $\lim_{\Delta y \to 0} \oiint_S \boldsymbol{D}.d\boldsymbol{S} = 0$
- It follows that $\boldsymbol{D}_1 \cdot A_o \boldsymbol{a}_y - \boldsymbol{D}_2 \cdot A_o \boldsymbol{a}_y = 0$ $\Longrightarrow$ $D_1^n = D_2^n$

or alternatively, $\boldsymbol{n}.(\boldsymbol{D}_1 - \boldsymbol{D}_2) = 0$

- But as $D_{1,2}^n = \varepsilon_{1,2} E_{1,2}^n$ $\Longrightarrow$ $E_1^n = \dfrac{\varepsilon_2}{\varepsilon_1} E_2^n$
- Normal components of the electric field are discontinuous across the interface

Finite Conductivity Case (Cont'd)

- Similarly, by applying Gauss's law for magnetic fields we get $\lim_{\Delta y \to 0} \oint\!\!\oint_S \boldsymbol{B}.d\boldsymbol{S} = 0$
- It follows that $\boldsymbol{B}_1 \cdot A_o \boldsymbol{a}_y - \boldsymbol{B}_2 \cdot A_o \boldsymbol{a}_y = 0 \Longrightarrow B_1^n = B_2^n$

or alternatively, $\boldsymbol{n}.(\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0$

- But as $B_{1,2}^n = \mu_{1,2} H_{1,2}^n \Longrightarrow H_1^n = \dfrac{\mu_2}{\mu_1} H_2^n$
- Normal components of the magnetic fields are discontinuous

Finite Conductivity Boundary Conditions

$\boldsymbol{n}\times(\boldsymbol{E}_1-\boldsymbol{E}_2)=\boldsymbol{0}$, no interface surface magnetic currents

$\boldsymbol{n}\times(\boldsymbol{H}_1-\boldsymbol{H}_2)=\boldsymbol{0}$, no interface surface electric currents

$\boldsymbol{n}.(\boldsymbol{D}_1-\boldsymbol{D}_2)=0$, no interface surface electric charges

$\Downarrow$

$E_1^n=\dfrac{\varepsilon_2}{\varepsilon_1}E_2^n$ discontinuous normal electric field

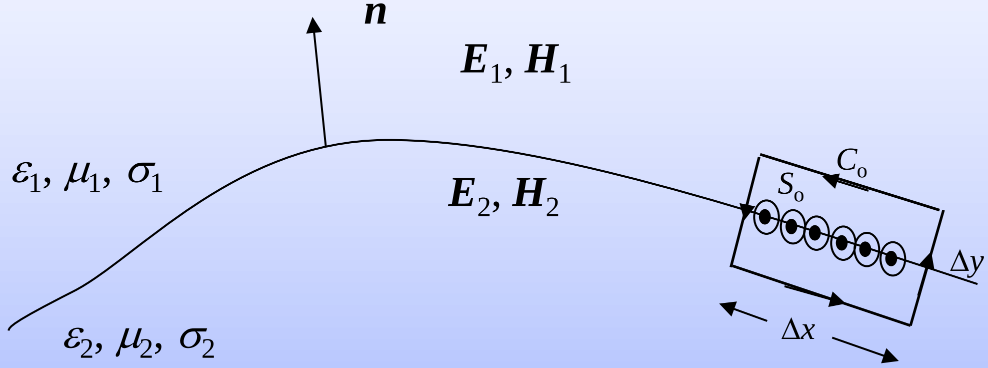
$\boldsymbol{n}.(\boldsymbol{B}_1-\boldsymbol{B}_2)=0$, no interface magnetic surface charges

$\Downarrow$

$H_1^n=\dfrac{\mu_2}{\mu_1}H_2^n$ discontinuous normal magnetic field

Boundary Conditions with Sources

- Boundary conditions must be changed to take into account the existence of surface currents and surface charges



- Applying the modified Ampere's law we get

$$\oint_C \boldsymbol{H}.\boldsymbol{dl} = \iint_S \boldsymbol{J}.\boldsymbol{dS} + \frac{\partial}{\partial t}\iint_S \boldsymbol{D}.\boldsymbol{dS}$$

- Taking the limit as $\Delta y \rightarrow 0$, each integral term gives

Boundary Conditions with Sources (Cont'd)

$$\lim_{\Delta y \to 0} \oint_C \boldsymbol{H}.d\boldsymbol{l} = (\boldsymbol{H}_2 - \boldsymbol{H}_1).\Delta x\, \boldsymbol{a}_x$$

$$\lim_{\Delta y \to 0} \iint_S \boldsymbol{D}.d\boldsymbol{S} = 0$$

$$\lim_{\Delta y \to 0} \iint_S \boldsymbol{J}.d\boldsymbol{S} = \lim_{\Delta y \to 0} \boldsymbol{J}.\Delta x \Delta y\, \boldsymbol{a}_z = \lim_{\Delta y \to 0} (\boldsymbol{J}\Delta y).\Delta x\, \boldsymbol{a}_z = \lim_{\Delta y \to 0} \boldsymbol{J}_s.\Delta x\, \boldsymbol{a}_z$$

$\boldsymbol{J}_s$ is the surface current density A/m

- It follows that $(\boldsymbol{H}_2 - \boldsymbol{H}_1).\Delta x\, \boldsymbol{a}_x = \boldsymbol{J}_s.\Delta x\, \boldsymbol{a}_z$

or alternatively $(\boldsymbol{H}_2 - \boldsymbol{H}_1).(\boldsymbol{a}_y \times \boldsymbol{a}_z) = \boldsymbol{J}_s.\boldsymbol{a}_z$

$$(\boldsymbol{a}_y \times (\boldsymbol{H}_1 - \boldsymbol{H}_2)).\boldsymbol{a}_z = \boldsymbol{J}_s.\boldsymbol{a}_z \Longrightarrow \boldsymbol{n} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_s$$

Boundary Conditions with Sources (Cont'd)

- Tangential components of the magnetic field intensity are discontinuous if surface electric current density $\boldsymbol{J}_s$ (A/m) exists
- If medium 2 is a perfect conductor, we have

$$\boldsymbol{n}\times\boldsymbol{H}_1=\boldsymbol{J}_s \Longrightarrow H_1^t=J_s$$

- Similarly, starting with Faraday's Law

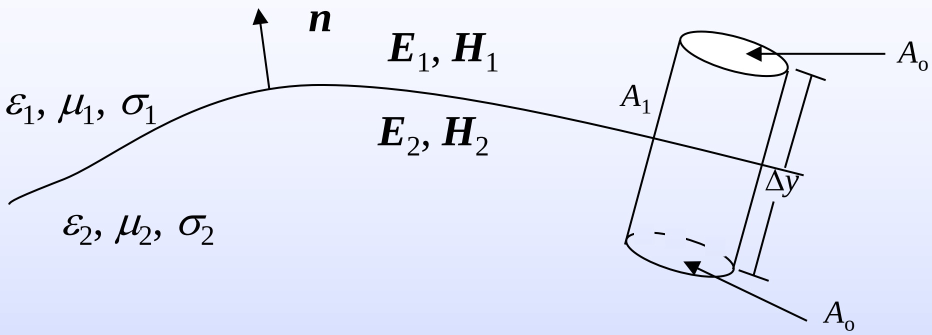
$$\oint_C \boldsymbol{E}.\boldsymbol{dl}=-\iint_S \boldsymbol{\mu}.\boldsymbol{dS}-\frac{\partial}{\partial t}\iint_S \boldsymbol{B}.\boldsymbol{dS}$$

We can reach $-\boldsymbol{n}\times(\boldsymbol{E}_1-\boldsymbol{E}_2)=\boldsymbol{\mu}_s$

- For a perfect conductor we have $-\boldsymbol{n}\times\boldsymbol{E}_1=\boldsymbol{\mu}_s$
- If no fictious magnetic current is assumed we have

$$-\boldsymbol{n}\times\boldsymbol{E}_1=0 \Longrightarrow E_1^t=0$$

Boundary Conditions with Sources (Cont'd)



- Applying Gauss's law for the shown cylinder we have

$$\oint\!\!\oint_S \boldsymbol{D}.\boldsymbol{dS} = \iiint_V q_{ev}\, dV \Longrightarrow \lim_{\Delta y \to 0} \oint\!\!\oint_{A_o} \boldsymbol{D}.\boldsymbol{dS} = \lim_{\Delta y \to 0} \iiint_{\boldsymbol{V}} q_{ev}\, dV$$

$$(\boldsymbol{D}_1 - \boldsymbol{D}_2).\, A_o \boldsymbol{n} = \lim_{\Delta y \to 0} (q_{ev} \Delta y)\, A_o = q_{es} A_o$$

or alternatively, $D_1^n - D_2^n = q_{es}$

- Normal components of the electric flux density are discontinuous by the amount of surface charge density

Boundary Conditions with Sources (Cont'd)

- If medium 2 is a perfect conductor, we have $D_1^n = q_{es}$
- Similarly, for the magnetic flux density we may show that $\boldsymbol{n}.(\boldsymbol{B}_1 - \boldsymbol{B}_2) = q_{ms}$
- For perfect conductors with no magnetic charges we have $B_2^n = B_1^n = 0$

Summary of Boundary Conditions

$$\boldsymbol{n}\times(\boldsymbol{H}_1-\boldsymbol{H}_2)=\boldsymbol{J}_s$$

$$-\boldsymbol{n}\times(\boldsymbol{E}_1-\boldsymbol{E}_2)=\boldsymbol{\mu}_s$$

$$\boldsymbol{n}.(\boldsymbol{D}_1-\boldsymbol{D}_2)=q_{es}$$

$$\boldsymbol{n}.(\boldsymbol{B}_1-\boldsymbol{B}_2)=q_{ms}$$

Time-Harmonic Electromagnetic Fields

- If sources are sinusoidal and the medium is linear then the fields everywhere are sinusoidal as well. The field at each point is characterized by its amplitude and phase (Phasor)

- Ex: $f(x,t) = 3.0\cos(\omega t - \beta x) = 3.0\,\mathrm{Re}(\exp(j(\omega t - \beta x)))$

$$f(x,t) = \mathrm{Re}(3.0\exp(-j\beta x)\exp(j\omega t))$$

$$f(x,t) = \mathrm{Re}(\tilde{f}\exp(j\omega t))$$

$$\tilde{f} = 3.0\exp(-j\beta x)$$

Time-Harmonic Electromagnetic Fields (Cont'd)

- Similarly, for all field quantities we may write

$$\boldsymbol{E}(x,y,z,t)=\mathrm{Re}(\tilde{\boldsymbol{E}}(x,y,z)\exp(j\omega t))$$

$$\boldsymbol{H}(x,y,z,t)=\mathrm{Re}(\tilde{\boldsymbol{H}}(x,y,z)\exp(j\omega t))$$

$$\boldsymbol{D}(x,y,z,t)=\mathrm{Re}(\tilde{\boldsymbol{D}}(x,y,z)\exp(j\omega t))$$

$$\boldsymbol{B}(x,y,z,t)=\mathrm{Re}(\tilde{\boldsymbol{B}}(x,y,z)\exp(j\omega t))$$

$$\boldsymbol{J}(x,y,z,t)=\mathrm{Re}(\tilde{\boldsymbol{J}}(x,y,z)\exp(j\omega t))$$

$$q(x,y,z,t)=\mathrm{Re}(\tilde{q}(x,y,z)\exp(j\omega t))$$

Time-Harmonic Electromagnetic Fields (Cont'd)

- Maxwell's equations for the time-harmonic case are obtained by replacing each time vector by its corresponding phasor vector and replacing $\partial/\partial t$ by $j\omega$
- Maxwell's equations in the integral form are given by

$$\oint\!\!\oint_S \tilde{\boldsymbol{D}}.\boldsymbol{dS} = \iiint_V \tilde{q}_{ev}\, dV = \tilde{Q}_{ev}$$

$$\oint\!\!\oint_S \tilde{\boldsymbol{B}}.\boldsymbol{dS} = \iiint_V \tilde{q}_{mv}\, dV = \tilde{Q}_{mv}$$

$$\oint_C \tilde{\boldsymbol{E}}.\boldsymbol{dl} = -\iint_S \tilde{\boldsymbol{\mu}}.\boldsymbol{dS} - j\omega \iint_S \tilde{\boldsymbol{B}}.\boldsymbol{dS}$$

$$\oint_C \tilde{\boldsymbol{H}}.\boldsymbol{dl} = \iint_S \tilde{\boldsymbol{J}}.\boldsymbol{dS} + j\omega \iint_S \tilde{\boldsymbol{D}}.\boldsymbol{dS}$$

$$\oint\!\!\oint_S \tilde{\boldsymbol{J}}.\boldsymbol{dS} = -j\omega \tilde{Q}_e$$

Time-Harmonic Electromagnetic Fields (Cont'd)

- Maxwell's equations in the differential form become

$$\nabla.\tilde{\boldsymbol{D}} = \tilde{q}_{ev}$$

$$\nabla.\tilde{\boldsymbol{B}} = \tilde{q}_{mv}$$

$$\nabla.\tilde{\boldsymbol{J}} = -j\omega\tilde{q}_{ev}$$

$$(\nabla\times\tilde{\boldsymbol{E}}) = -\tilde{\boldsymbol{\mu}} - j\omega\tilde{\boldsymbol{B}}$$

$$(\nabla\times\tilde{\boldsymbol{H}}) = \tilde{\boldsymbol{J}} + j\omega\tilde{\boldsymbol{D}}$$

- Same boundary conditions apply

Energy and Power

- We would like to derive equations governing EM energy and power
- Starting with Maxwell's equations

$$(\nabla \times \boldsymbol{E}) = -\boldsymbol{\mu}_i - \frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\mu}_i - \boldsymbol{\mu}_d \qquad (.\boldsymbol{H})$$

$$(\nabla \times \boldsymbol{H}) = \boldsymbol{J}_i + \boldsymbol{J}_c + \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J}_i + \boldsymbol{J}_c + \boldsymbol{J}_d \qquad (.\boldsymbol{E})$$

Subtracting we get

$$\boldsymbol{H} \bullet (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \bullet (\nabla \times \boldsymbol{H}) = -\boldsymbol{H} \bullet (\boldsymbol{\mu}_i + \boldsymbol{\mu}_d) - \boldsymbol{E} \bullet (\boldsymbol{J}_i + \boldsymbol{J}_c + \boldsymbol{J}_d)$$

or alternatively,

$$\nabla \bullet (\boldsymbol{E} \times \boldsymbol{H}) = -\boldsymbol{H} \bullet (\boldsymbol{\mu}_i + \boldsymbol{\mu}_d) - \boldsymbol{E} \bullet (\boldsymbol{J}_i + \boldsymbol{J}_c + \boldsymbol{J}_d)$$

Energy and Power (Cont'd)

- Integrating over the volume of interest

$$\iiint_V \nabla \bullet (\boldsymbol{E} \times \boldsymbol{H}) dV = -\iiint_V \boldsymbol{H} \bullet (\boldsymbol{\mu}_i + \boldsymbol{\mu}_d) dV - \iiint_V \boldsymbol{E} \bullet (\boldsymbol{J}_i + \boldsymbol{J}_c + \boldsymbol{J}_d) dV$$

- Utilizing the divergence theorem, we get

$$\oiint_S (\boldsymbol{E} \times \boldsymbol{H}) \bullet \boldsymbol{dS} + \iiint_V \boldsymbol{H} \bullet (\boldsymbol{\mu}_i + \boldsymbol{\mu}_d) dV + \iiint_V \boldsymbol{E} \bullet (\boldsymbol{J}_i + \boldsymbol{J}_c + \boldsymbol{J}_d) dV = 0$$

- Explanation of different terms

$\boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{H}$ is the Poynting vector (W/m^2)

$\boldsymbol{P}_o = \oiint_S \boldsymbol{P}.\boldsymbol{dS}$ is the power flowing out of the surface S

$\boldsymbol{P}_S = -\iiint_V (\boldsymbol{H} \bullet \boldsymbol{\mu}_i + \boldsymbol{E} \bullet \boldsymbol{J}_i) \, dV$ is the supplied power (W)

Energy and Power (Cont'd)

$$P_d = \iiint_V \boldsymbol{E} \bullet \boldsymbol{J}_c \, dV = \iiint_V \sigma \boldsymbol{E} \bullet \boldsymbol{E} dV = \iiint_V \sigma \left|\boldsymbol{E}\right|^2 dV$$

=dissipated power (W)

$$P_m = \iiint_V \boldsymbol{H} \bullet \boldsymbol{\mu}_d \, dV = \iiint_V \boldsymbol{H} \bullet \frac{\partial \boldsymbol{B}}{\partial t} dV = \iiint_V \mu \boldsymbol{H} \bullet \frac{\partial \boldsymbol{H}}{\partial t} dV$$

$$P_m = \frac{\partial}{\partial t} \iiint_V \frac{1}{2} \mu \left|\boldsymbol{H}\right|^2 \boldsymbol{dV} = \frac{\partial}{\partial t} W_m = \text{magnetic power}$$

$W_m =$ magnetic energy

$$P_e = \iiint_V \boldsymbol{E} \bullet \boldsymbol{J}_d \, dV = \iiint_V \boldsymbol{E} \bullet \frac{\partial \boldsymbol{D}}{\partial t} dV = \iiint_V \varepsilon \boldsymbol{E} \bullet \frac{\partial \boldsymbol{E}}{\partial t} dV$$

$$P_e = \frac{\partial}{\partial t} \iiint_V \frac{1}{2} \varepsilon \left|\boldsymbol{E}\right|^2 \boldsymbol{dV} = \frac{\partial}{\partial t} W_e = \text{electric power}$$

$W_e =$ electric energy

$$\boxed{P_s = P_o + P_d + \partial (W_e + W_m) / \partial t}$$ conservation of EM energy