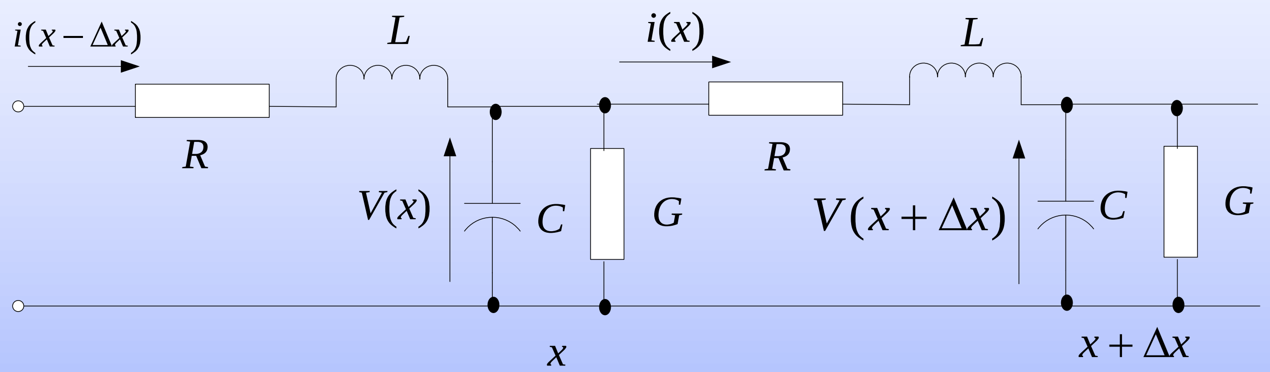
EE757
Numerical Techniques in Electromagnetics
Lecture 4

1D TLM

- We establish a one-to-one mapping between 1D wave equations and a network of transmission lines



- C is the capacitance of a section of length Δx, $C=C_d\Delta x$
- L is the inductance of a section of length Δx, $L=L_d\Delta x$
- R and G represent series resistance and shunt conductance, respectively

1D TLM (Cont'd)

- Applying KVL and KCL, we get

$$\Delta x \frac{\partial V}{\partial x} = -L\frac{\partial i}{\partial t} - iR \quad \text{and} \quad \Delta x \frac{\partial i}{\partial x} = -C\frac{\partial V}{\partial t} - GV$$

- Differentiating the first equation w.r.t *t* and the second equation w.r.t. *x* we get

$$\frac{\partial^2 i}{\partial x^2} = \frac{GR}{(\Delta x)^2} i + \frac{GL+RC}{(\Delta x)^2}\frac{\partial i}{\partial t} + \frac{LC}{(\Delta x)^2}\frac{\partial^2 i}{\partial t^2}$$

- Similarly, we can show that

$$\frac{\partial^2 V}{\partial x^2} = \frac{GR}{(\Delta x)^2} V + \frac{GL+RC}{(\Delta x)^2}\frac{\partial V}{\partial t} + \frac{LC}{(\Delta x)^2}\frac{\partial^2 V}{\partial t^2}$$

Correspondence with Maxwell's Equations

- For a 1D source-free problem, the fields depend only on one coordinate (say x)

- Maxwell's equations are given by

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \quad \text{and} \quad -\frac{\partial H_z}{\partial x} = J_{cy} + \frac{\partial D_y}{\partial t}$$

- Similarly, we obtain $\displaystyle \frac{\partial^2 E_y}{\partial x^2} = \mu\sigma \frac{\partial E_y}{\partial t} + \mu\varepsilon \frac{\partial^2 E_y}{\partial t^2}$

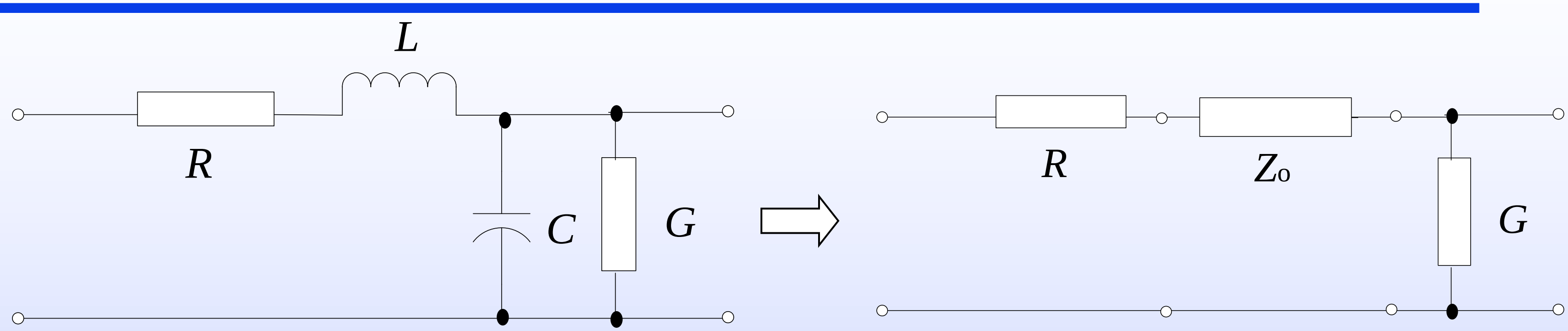
- Comparing with case R=0, $\displaystyle \frac{\partial^2 V}{\partial x^2} = \frac{GL}{(\Delta x)^2} \frac{\partial V}{\partial t} + \frac{LC}{(\Delta x)^2} \frac{\partial^2 V}{\partial t^2}$

We obtain the one-to-one correspondence

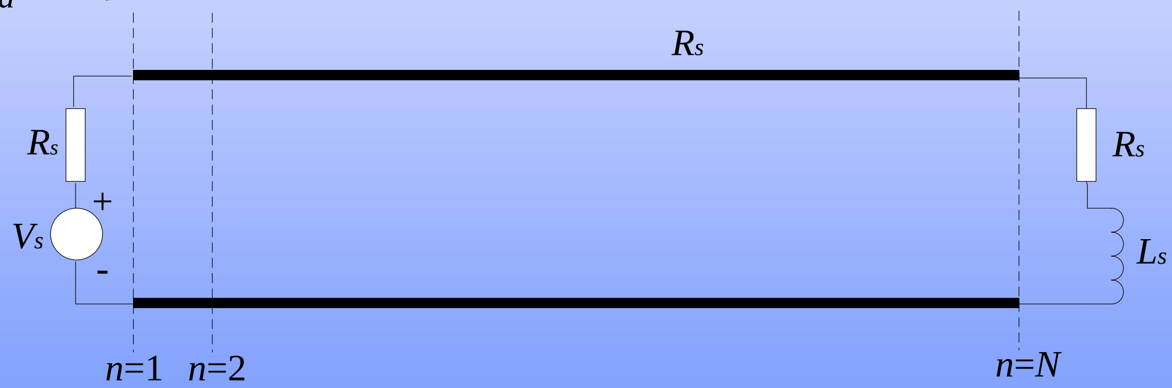
$V \leftrightarrow E_y$, $\mu \leftrightarrow (L/\Delta x)$, $\varepsilon \leftrightarrow (C/\Delta x)$, $\sigma \leftrightarrow (G/\Delta x)$

- Solving the discretized TLM network obtains a solution of the corresponding EM problem

Solution of the TLM Network



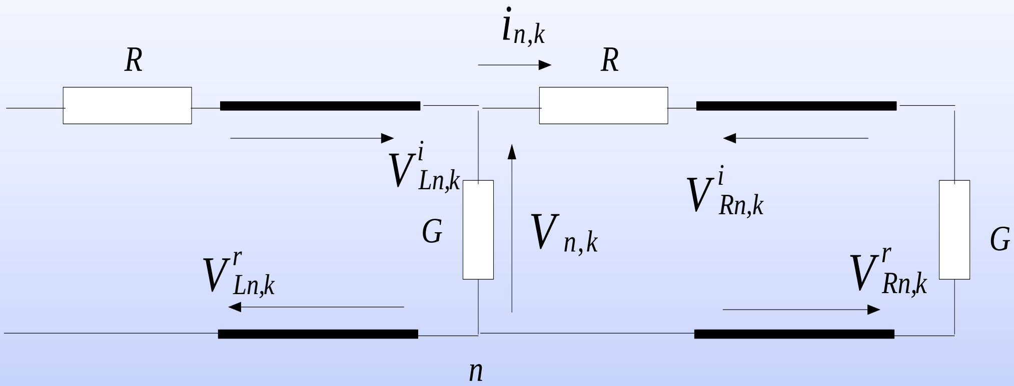
$$Z_o = \sqrt{\frac{L_d}{C_d}} = \sqrt{\frac{L}{C}} \quad , \quad \Delta t = \frac{\Delta x}{u} = \Delta x\sqrt{L_d\, C_d} = \sqrt{LC}$$

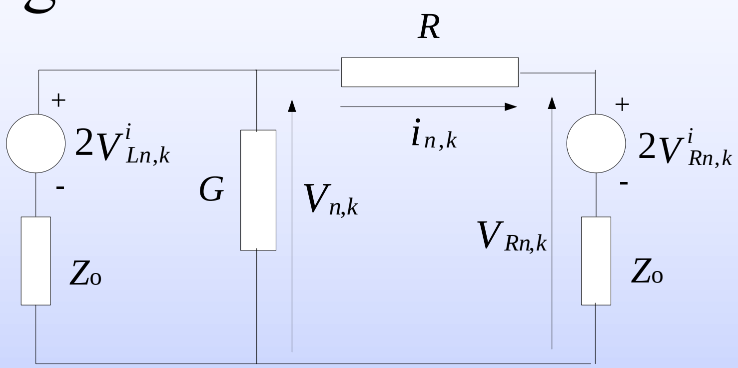


- Consider the following model with N sections
- Node n is between sections (n-1) and n, $1<n<(N+1)$

Solution of a TLM Network (Cont'd)

- Utilizing Thevenin's equivalent we get





- Using Superposition or Milliman's theorem, we get

$$V_{n,k} = \frac{\dfrac{2V^i_{Ln,k}}{Z_o} + \dfrac{2V^i_{Rn,k}}{Z_o}}{\dfrac{1}{Z_o} + G + \dfrac{1}{Z_o + R}} \quad \Rightarrow \quad i_{n,k} = \frac{V_{n,k} - 2V^i_{Rn,k}}{Z_o + R}$$

$$V_{Rn,k} = 2V^i_{Rn,k} + i_{n,k} Z_o \quad , \quad V_{Ln,k} = V_{n,k}$$

Solution of a TLM Network (Cont'd)

- Using all the voltages and current, we carry out the two fundamental steps of the TLM method:

- Scattering: Evaluate the reflected waves

$$V^r_{Rn,k} = V_{Rn,k} - V^i_{Rn,k} \;,\quad V^r_{Ln,k} = V_{Ln,k} - V^i_{Ln,k}$$

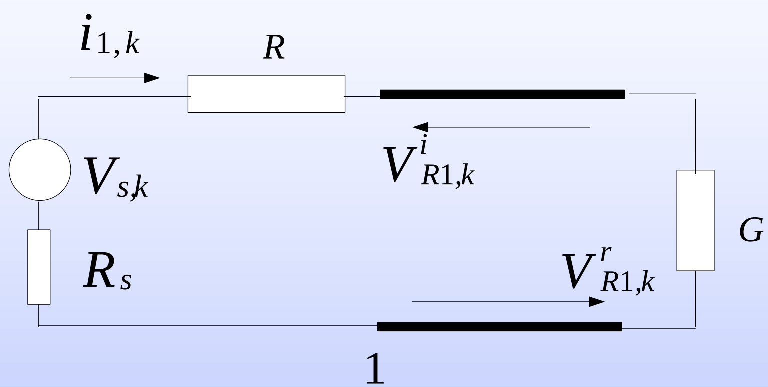
- Connection: determine the incident waves at the (k+1) time step ⇨ reflected waves become incident waves on neighboring nodes at the next time step

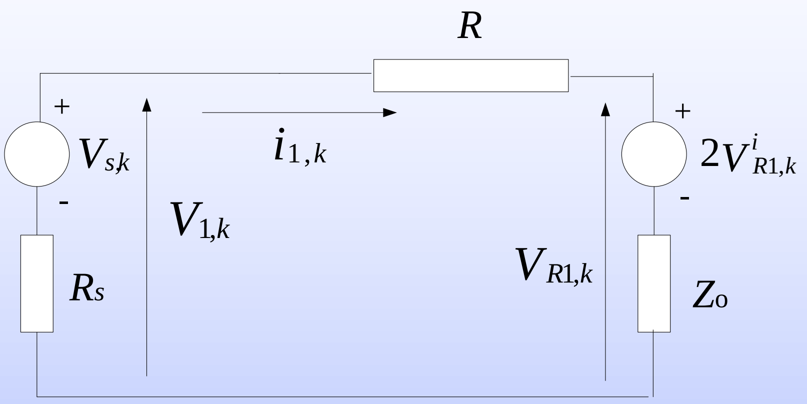
$$V^i_{Ln,(k+1)} = V^r_{R(n\text{-}1),k}$$

$$V^i_{Rn,(k+1)} = V^r_{L(n+1),k}$$

Solution of a TLM Network (Cont'd)

- For the source node we have



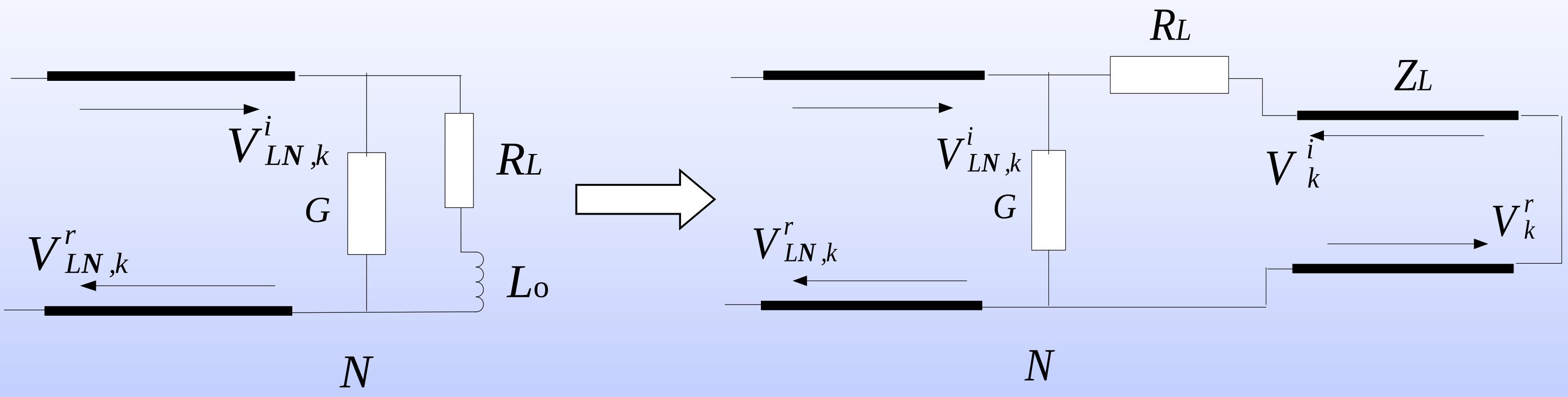


$$i_{1,k} = \frac{V_{s,k} - 2V^i_{R1,k}}{R + R_s + Z_o} \quad , \quad V_{R1,k} = 2V^i_{R1,k} + i_{1,k} Z_o$$

$$V^r_{R1,k} = V_{R1,k} - V^i_{R1,k} \quad , \quad V^i_{R1,k+1} = V^r_{L2,k}$$

Solution of the TLM Network (Cont'd)

- For the load node we have



- Note that $Z_L = L_o/(\Delta t/2)$ ⟹ synchronization is preserved

- Derive the scattering and connection relationships for the load node

General Steps

- Given the parameters of the electromagnetic problem determine the parameters of the TLM network (Source, load, time step, L, C)

- Initialize incident impulses for all sections (left and right)

- Repeat for all time steps

  * Evaluate intermediate quantities (voltages and currents)

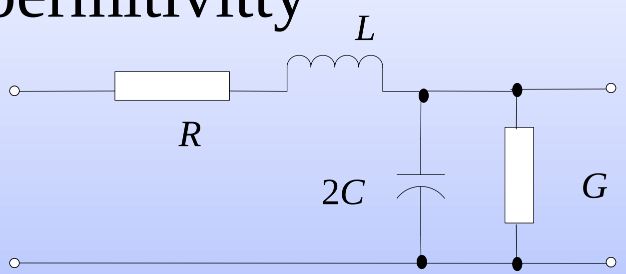
  * Evaluate reflected waves (scattering)

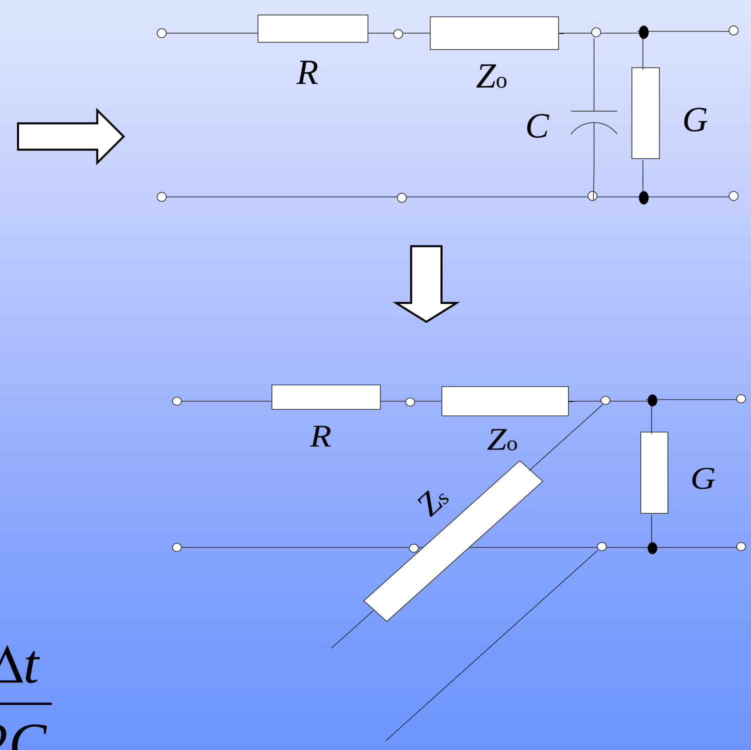
  * Obtain incident waves at the next time step from the reflected impulses at the current time step (Connection)

  end

The Nonhomogenous Case

- To maintain synchronization, stubs should be used
- Example: a section of length Δx with double the permitivitty



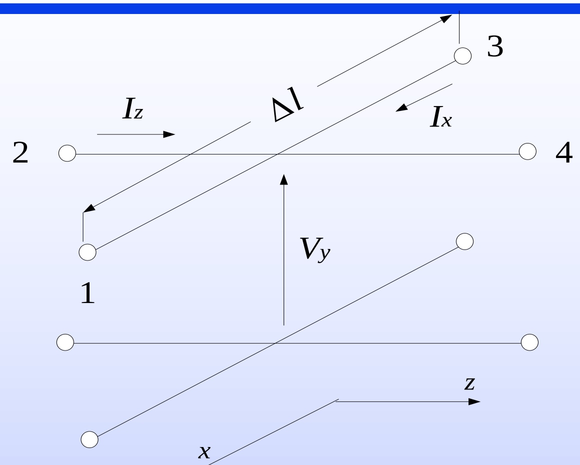


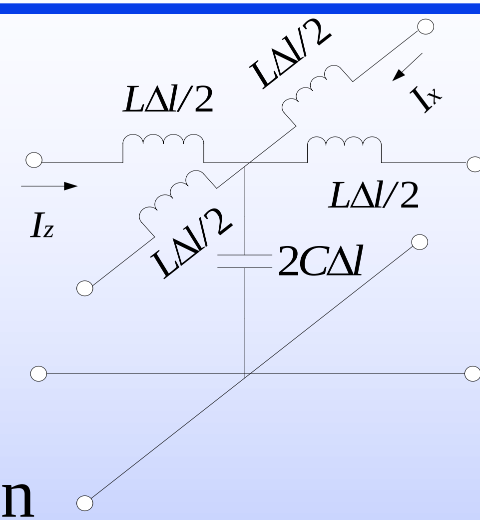
$$Z_o = \sqrt{\frac{L}{C}} \quad , \quad u = \frac{\Delta x}{\sqrt{LC}}$$

$$\Delta t = \frac{\Delta x}{u} = \sqrt{LC}$$

For the stub we have $Z_s = \dfrac{\Delta t}{2C}$

2D TLM using Shunt Nodes





- Using KVL along the *x* direction

$$V_y(x+\Delta x) = V_y(x) - L\Delta x \partial I_x/\partial t \Longrightarrow \boxed{\frac{\partial V_y}{\partial x} = -L\frac{\partial I_x}{\partial t}}$$

- Using KVL along the z direction we have

$$V_y(z+\Delta z) = V_y(z) - L\Delta z \partial I_z/\partial t \Longrightarrow \boxed{\frac{\partial V_y}{\partial z} = -L\frac{\partial I_z}{\partial t}}$$

- Using KCL we have $I_z(z) + I_x(x) = I_z(z+\Delta z) + I_x(x+\Delta x) + 2C\Delta l\dfrac{\partial V_y}{\partial t}$

$$-\frac{\partial I_z}{\partial z}\Delta l - \frac{\partial I_x}{\partial x}\Delta l = 2C\Delta l\frac{\partial V_y}{\partial t} \Longrightarrow \boxed{\frac{\partial I_z}{\partial z} + \frac{\partial I_x}{\partial x} = -2C\frac{\partial V_y}{\partial t}}$$

2D TLM using Shunt Nodes (Cont'd)

- Combining these equations we get $\dfrac{\partial^2 V_y}{\partial x^2}+\dfrac{\partial^2 V_y}{\partial z^2}=2LC\dfrac{\partial^2 V_y}{\partial t^2}$
- Maxwell's equations for the lossless TM_y case are

$$\frac{\partial E_y}{\partial z}=\mu\frac{\partial H_x}{\partial t},\quad \frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=\varepsilon\frac{\partial E_y}{\partial t},\quad \frac{\partial E_y}{\partial x}=-\mu\frac{\partial H_z}{\partial t}$$

- Combining these equations, we get $\dfrac{\partial^2 E_y}{\partial x^2}+\dfrac{\partial^2 E_y}{\partial z^2}=\mu\varepsilon\dfrac{\partial^2 E_y}{\partial t^2}$
- We now establish the equivalences

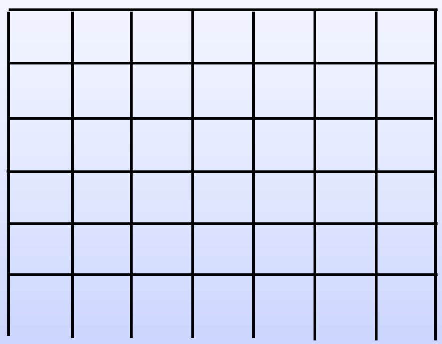
$$E_y\leftrightarrow V_y,\ H_z\leftrightarrow I_x,\ H_x\leftrightarrow -I_z,\ \mu\leftrightarrow L,\ \varepsilon\leftrightarrow 2C$$

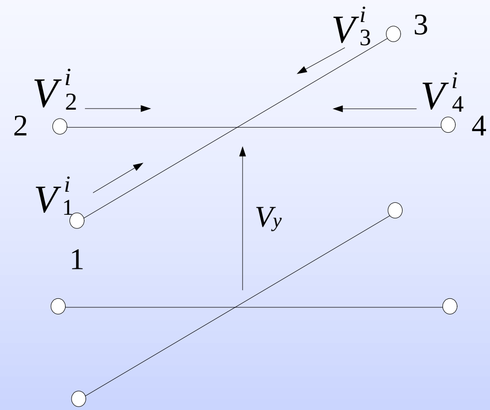
- network velocity=medium velocity $1/\sqrt{\mu\varepsilon}=1/\sqrt{2CL}$
- The link velocity v_l is $v_l=1/\sqrt{LC}=\sqrt{2}/\sqrt{\mu\varepsilon}=\sqrt{2}\,v_n$
- The link characteristic impedance is

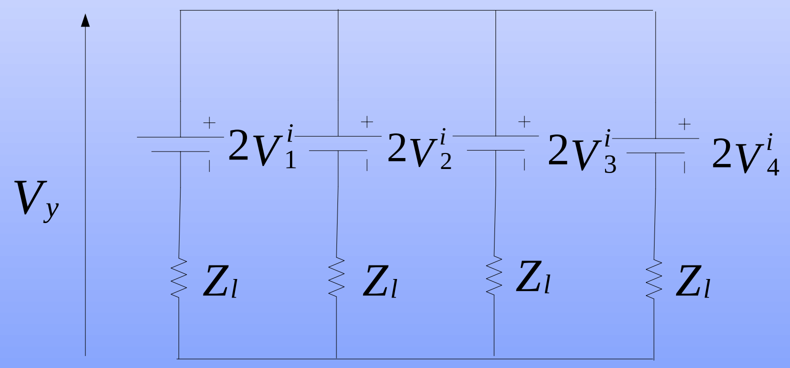
$$Z_l=\sqrt{L/C}=\sqrt{2}\sqrt{\mu/\varepsilon}=\sqrt{2}\,Z_n$$

Scattering in Shunt TLM Nodes (Lossless Case)

- The computational domain is filled with TLM cells







- We replace each section by its Thevenin's equivalent
- Using Superposition we have $V_y = 0.5(V_1^i + V_2^i + V_3^i + V_4^i)$

Scattering in Shunt TLM Nodes (Cont'd)

- It follows that we have

$$V_1^r = V_y - V_1^i = 0.5(-V_1^i + V_2^i + V_3^i + V_4^i)$$

$$V_2^r = V_y - V_2^i = 0.5(V_1^i - V_2^i + V_3^i + V_4^i)$$

$$V_3^r = V_y - V_3^i = 0.5(V_1^i + V_2^i - V_3^i + V_4^i)$$

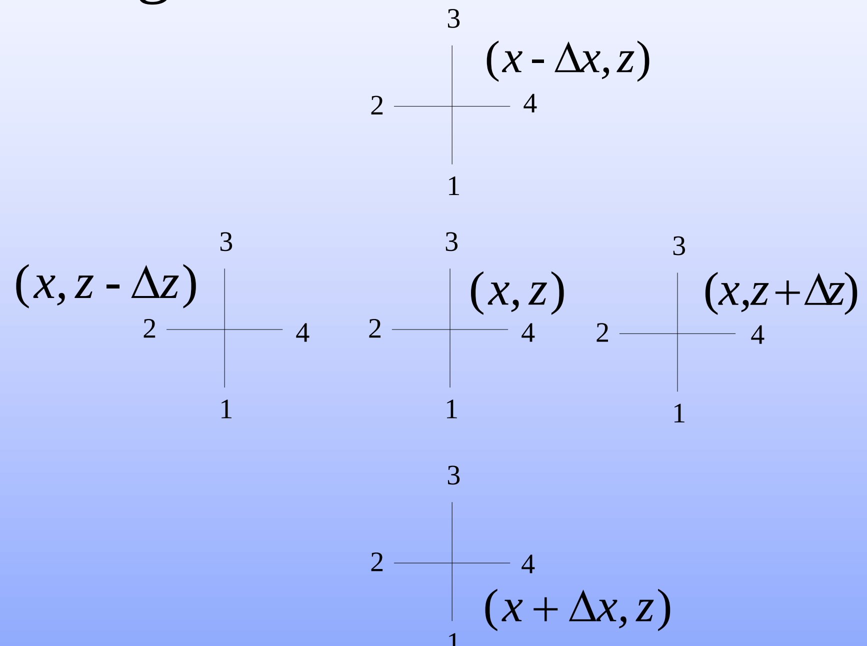
$$V_4^r = V_y - V_4^i = 0.5(V_1^i + V_2^i + V_3^i - V_4^i)$$

- Or in matrix form

$$\begin{bmatrix} V_1^r \\ V_2^r \\ V_3^r \\ V_4^r \end{bmatrix} = 0.5 \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} V_1^i \\ V_2^i \\ V_3^i \\ V_4^i \end{bmatrix}$$

Connection in Shunt TLM Nodes

- Reflected impulses at the *k*th time step become incident on neighboring nodes at the *k*+1 time step



$$\left(V_1^i(x,z)\right)_{k+1} = \left(V_3^r(x+\Delta x,z)\right)_k \quad , \quad \left(V_2^i(x,z)\right)_{k+1} = \left(V_4^r(x,z-\Delta z)\right)_k$$

$$\left(V_3^i(x,z)\right)_{k+1} = \left(V_1^r(x-\Delta x,z)\right)_k \quad , \quad \left(V_4^i(x,z)\right)_{k+1} = \left(V_2^r(x,z+\Delta z)\right)_k$$

Modeling of Free Space

- To model free space, the following two conditions must be satisfied

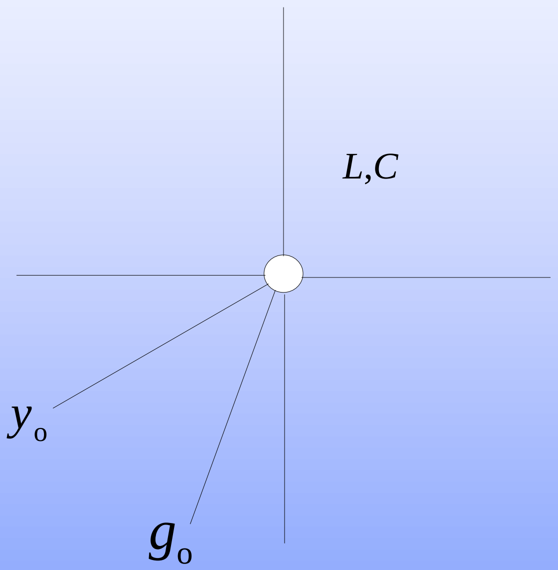
$$\frac{1}{\sqrt{2LC}} = \frac{1}{\sqrt{\mu_o \varepsilon_o}} \Longrightarrow \sqrt{2LC} = \frac{1}{v_o} = 3.0\text{e}8\ m/s$$

$$\sqrt{\frac{L}{2C}} = \sqrt{\frac{\mu_o}{\varepsilon_o}} = \eta_o = 377\Omega$$

- It follows that $L = \eta_o / v_o = \mu_o$ and $C = 1/(2\eta_o v_o) = \varepsilon_o / 2$
- v_L = link velocity= $1/\sqrt{LC} = v_o\sqrt{2}$ m/s
- Z_L= link characteristic impedance= $\sqrt{L/C} = \sqrt{2}\,\eta_o \Omega$
- Unit time step $\Delta t = \Delta l / v_L = \Delta l / (v_o\sqrt{2})$ sec

Modeling of a General Lossy Medium

- Additional losses and capacitances are modeled by matched shunt stubs and open ended shunt stubs, respectively



- y_o : normalized admittance of the $\Delta l/2$ shunt capacitance stub
- g_o : normalized admittance of the $\Delta l/2$ shunt loss stub

Modeling of a General Lossy Medium (Cont'd)

- Normalizing admittance is $Y_L = \sqrt{C/L}$

$$Y_\text{o} = y_\text{o}\sqrt{C/L} \quad \text{and} \quad G_\text{o} = g_\text{o}\sqrt{C/L}$$

- We choose C_s and L_s for the permitivitty stub to be

$C_s = Cy_o$ and $L_s = L/y_o$ $\Longrightarrow 1/\sqrt{L_s C_s} = 1/\sqrt{LC}$, $\sqrt{\dfrac{C_s}{L_s}} = Y_\text{o} = y_\text{o} Y_L$

- Similarly for the loss stub we have

$C_g = Cg_o$ and $L_g = L/g_o$ $\Longrightarrow 1/\sqrt{L_g C_g} = 1/\sqrt{LC}$, $\sqrt{\dfrac{C_g}{L_g}} = G_\text{o} = g_\text{o} Y_L$

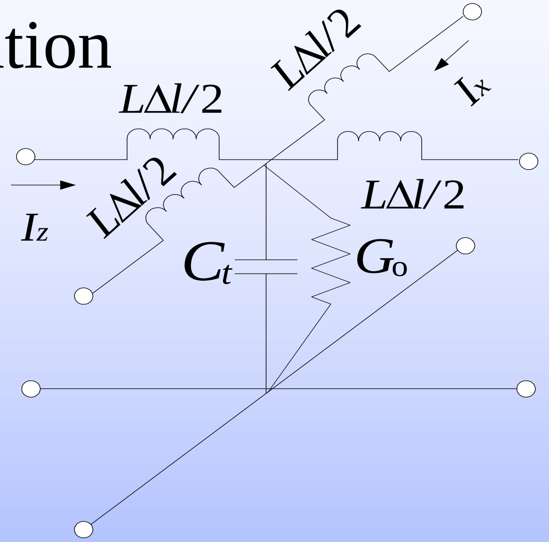
- The permitivitty stub represents a capacitance of value

$C_s \Delta l/2 = Cy_o \Delta l/2$ $\Longrightarrow$ Total cell capacitance= $Cy_o \Delta l/2 + 2C$

$$C_t = 2C\Delta l\left(1 + \frac{y_\text{o}}{4}\right)$$

Modeling of a General Lossy Medium (Cont'd)

- For small cell size, the cell now has the equivalent lumped presentation



- Equations governing this lossy node are

$$\partial V_y / \partial x = -L \partial I_x / \partial t \quad \text{(KVL in the } x \text{ direction)}$$

$$\partial V_y / \partial z = -L \partial I_z / \partial t \quad \text{(KVL in the } z \text{ direction)}$$

$$\frac{\partial I_z}{\partial z} + \frac{\partial I_x}{\partial x} = -2C(1 + y_o / 4)\frac{\partial V_y}{\partial t} - \frac{g_o \sqrt{C/L}}{\Delta l} V_y \quad \text{(KCL)}$$

Correspondence with Maxwell's Equations

- Maxwell's equations for the lossy 2D TM$_y$ case are

$$\partial E_y / \partial x = -\mu \partial H_z / \partial t$$

$$\partial E_y / \partial z = \mu \partial H_x / \partial t$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t} + \sigma E_y$$

- It follows that we can establish the 1-1 correspondence

$$E_y \leftrightarrow V_y, \quad H_z \leftrightarrow I_x, \quad H_x \leftrightarrow -I_z$$

$$\mu \leftrightarrow L, \quad \varepsilon \leftrightarrow 2C\left(1+\frac{y_o}{4}\right), \quad \sigma \leftrightarrow \frac{g_o\sqrt{C/L}}{\Delta l}$$

Some Practical Points

- *L* and *C* are usually selected to model free space with $L=\mu_o$ and $C=\varepsilon_o/2$
- y_o is adjusted at each node to model the local permitivitty

$$\varepsilon = 2C\left(1+\frac{y_o}{4}\right) \Longrightarrow \varepsilon_r = \left(1+\frac{y_o}{4}\right) \Longrightarrow y_o = 4.0(\varepsilon_r - 1)$$

- g_o is adjusted at each node to model the local conductivity

$$\sigma = \frac{g_o\sqrt{C/L}}{\Delta l}$$

References

P.B. Johns, The solution of inhomogeneous waveguide problems using a transmission-line matrix, IEEE Transactions MTT-22, 209-215, 1974

W.J.R. Hoefer, “The transmission-line matrix method-theory and applications,” *IEEE Trans. Microwave Theory Tech*., vol. MTT-33, pp. 882-893, Oct. 1985.