Lecture 4:Circuit Theory (4)

why phasor analysis. Complex numbers and phasors.

Types of Analysis

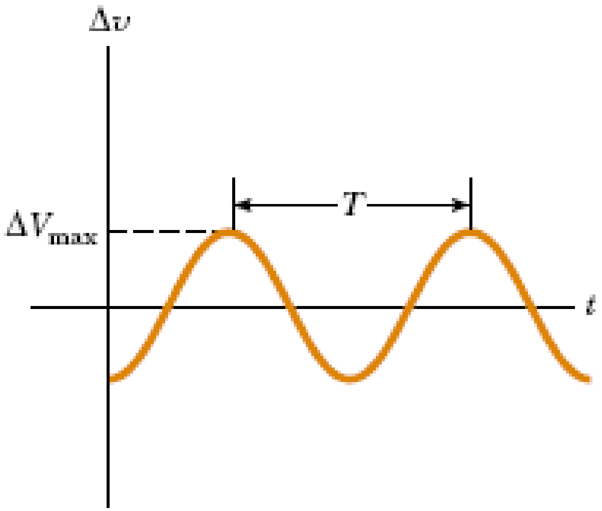
DC Analysis: all sources are constants with time resulting in constant currents and voltages

Transient Analysis: Sources are changing with time resulting in voltages and currents that also vary with time

AC Analysis: is a special case of transient analysis where the sources are sinusoidal. For a linear circuit, this results in currents and voltages that are also sinusoidal in time but with different amplitude and phase. The voltages and currents can thus be represented by their amplitudes and phases (phasors)

AC Signals

An AC power supply provides voltage difference that is alternating with time.



This alternation is usually sinusoidal in the form:

$$v = V_{\max} \cos\left(\omega t + \varphi_V\right)$$

where: $V_{\max} \equiv$ amplitude (V), $\varphi_V \equiv$ phase reference (rad),

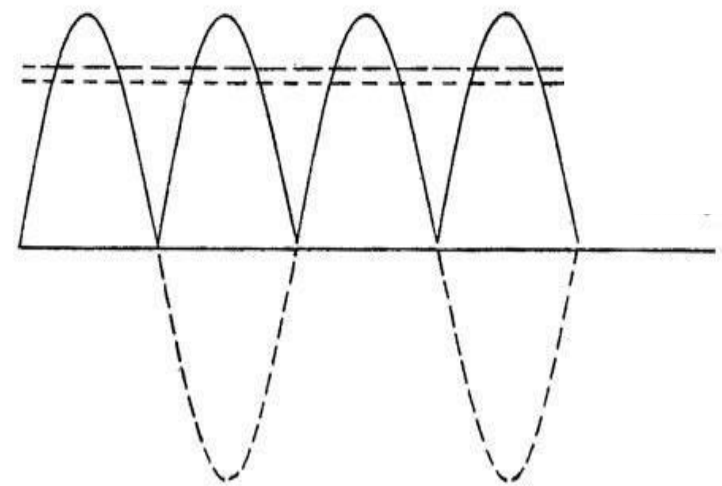
$\omega \equiv$ angular frequency (rad/s) $= 2\pi/T = 2\pi f$,

$T \equiv$ periodic time (s), $f \equiv$ frequency (Hz) $= 1/T$

What is the average (mean value) of an AC signal?

Root Mean Square (RMS)

The strength of the AC signal is measured using either the maximum value or the root-mean-square (rms), which is known as the effective value:



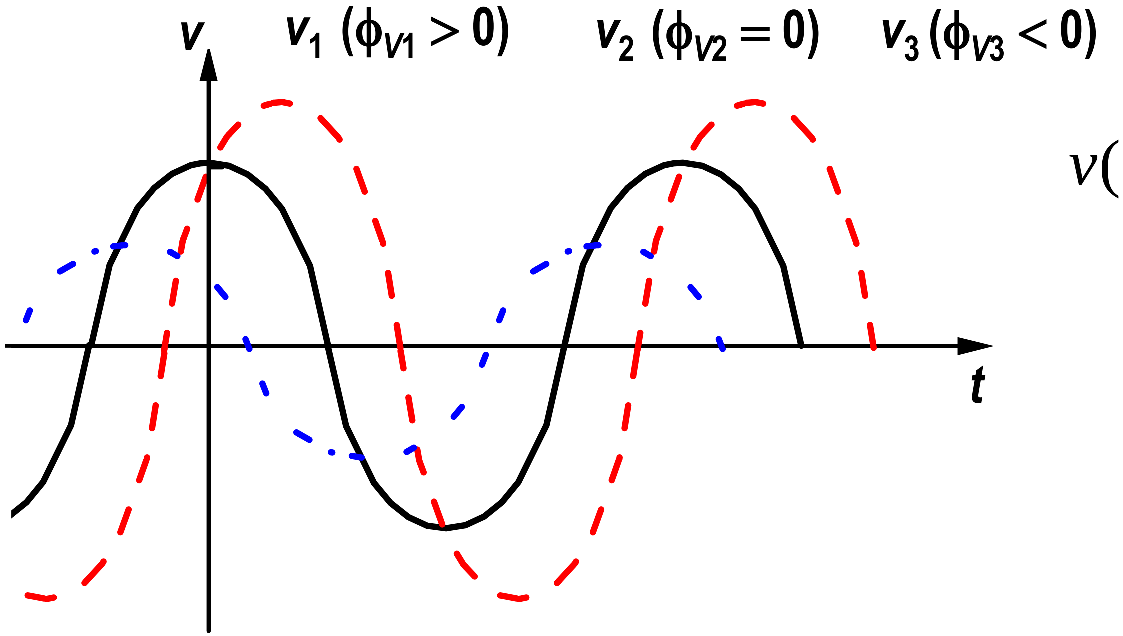
$$i^2 = I_{\max}^2 \cos^2(\omega t + \varphi_I) = \frac{I_{\max}^2}{2}\left[\cos(2\omega t + 2\varphi_I) + 1\right]$$

$$\overline{i^2} = \frac{1}{T}\int_0^T \frac{I_{\max}^2}{2}\left[\cos(2\omega t + 2\varphi_I) + 1\right] dt$$

$$= \frac{1}{T}\frac{I_{\max}^2}{2}\underbrace{\int_0^T \cos(2\omega t + 2\varphi_I)\, dt}_{=0} + \frac{1}{T}\frac{I_{\max}^2}{2}\underbrace{\int_0^T dt}_{=T} = \frac{I_{\max}^2}{2}$$

$$\therefore I_{rms} = \sqrt{\overline{i^2}} = \frac{I_{\max}}{\sqrt{2}} = 0.707\ I_{\max},\quad \underline{\text{Similarly:}}\ V_{rms} = \sqrt{\overline{v^2}} = \frac{V_{\max}}{\sqrt{2}} = 0.707\ V_{\max}$$

Time Representation



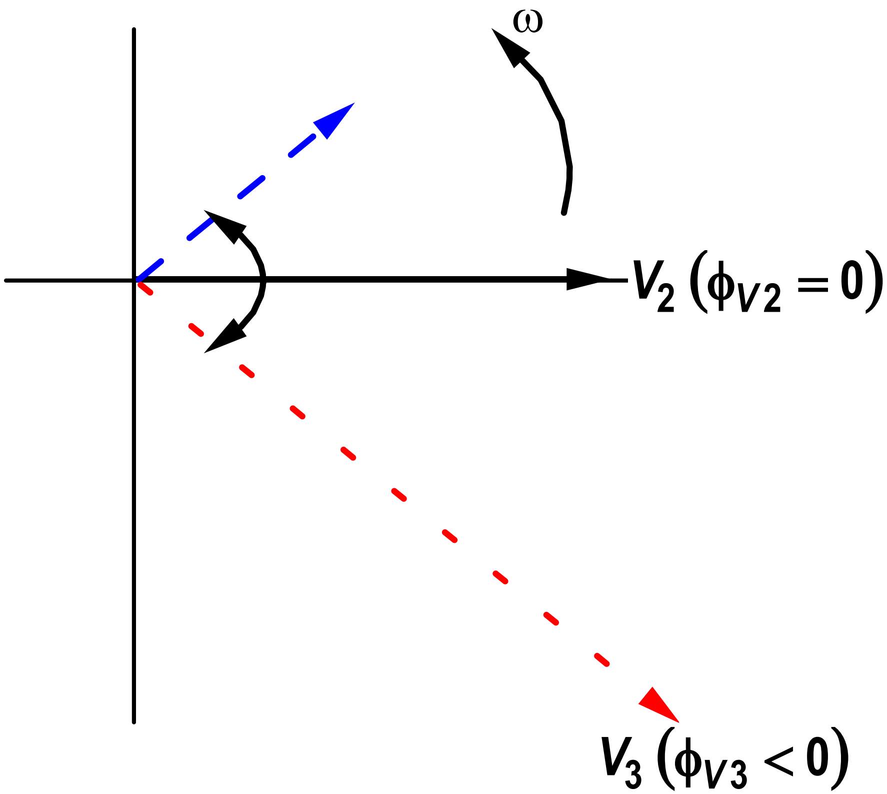
$$v(t) = \Re e\left(e^{j\omega t}\ V\right)$$

$$v = V_{\max}\ \cos\ \left(\omega t + \varphi_V\right) = f\left(t\right)$$

$V_{\max} \equiv$ amplitude of the sinusoidal signal

$\varphi_V \equiv$ phase reference (shift) of the sinusoidal signal

Phasor Representation



$$V = V_{\max}\ e^{j\varphi_V} \neq f\left(t\right)$$

$$V_{\max} \equiv \text{magnitude of the phasor} = \left| V \right|$$

$$\varphi_V \equiv \text{phase angle of the phasor}$$

Differentiation and Integration

Differentiation w.r.t. Time

$$y = \Re e\left[\, e^{j\omega t}\ Y \,\right] = \frac{dx}{dt}$$

$$= \frac{d}{dt}\left[\, \Re e\left(e^{j\omega t}\ X\right) \,\right]$$

$$= \Re e\left[\, \frac{d}{dt}\left(e^{j\omega t}\ X\right) \,\right]$$

$$= \Re e\left[\, X\, \frac{d}{dt}\left(e^{j\omega t}\right) \,\right]$$

$$= \Re e\left[\, X\ j\omega\ e^{j\omega t} \,\right]$$

$$= \Re e\left[\, e^{j\omega t}\ \left(j\omega\, X\right) \,\right]$$

$$\therefore y = \frac{dx}{dt} \iff Y = j\omega\, X$$

Integration w.r.t Time

$$y = \Re e\left[\, e^{j\omega t}\ Y \,\right] = \int x\ dt$$

$$= \int \Re e\ \left(e^{j\omega t}\ X\right)\ dt$$

$$= \Re e\left[\, \int \left(e^{j\omega t}\ X\right)\ dt \,\right]$$

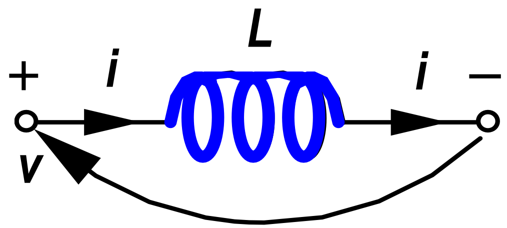
$$= \Re e\left[\, X\ \int \left(e^{j\omega t}\right)\ dt \,\right]$$

$$= \Re e\left[\, X\ \frac{1}{j\omega}\ e^{j\omega t} \,\right]$$

$$= \Re e\left[\, e^{j\omega t}\ \left(\frac{X}{j\omega}\right) \,\right]$$

$$\therefore y = \int x\ dt \iff Y = \frac{X}{j\omega}$$

Inductor



Impedance

$$v = L\ \frac{di}{dt}$$

$$V = L\ \left(j\omega\ I \right)$$

$$= \left(j\omega L \right)\ I$$

$$\equiv Z\ I$$

Admittance

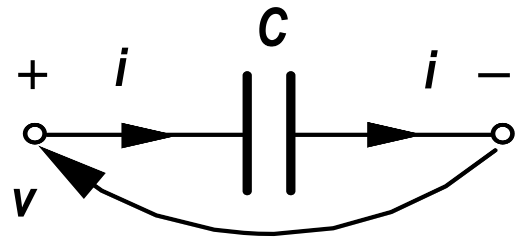
$$i = \frac{1}{L}\ \int\ v\ dt$$

$$I = \frac{1}{L}\ \left(V / j\omega \right)$$

$$= \left[\ 1/\left(j\omega L \right)\ \right]\ V$$

$$\equiv\ Y\ V$$

Capacitor



Impedance

$$v = \frac{1}{C} \int i \; dt$$

$$V = \frac{1}{C} \left(I / j\omega \right)$$

$$= \left[1 / \left(j\omega C \right) \right] I$$

$$\equiv Z \; I$$

Admittance

$$i = C \; \frac{dv}{dt}$$

$$I = C \left(j\omega \; V \right)$$

$$= \left(j\omega C \right) V$$

$$\equiv Y \; V$$

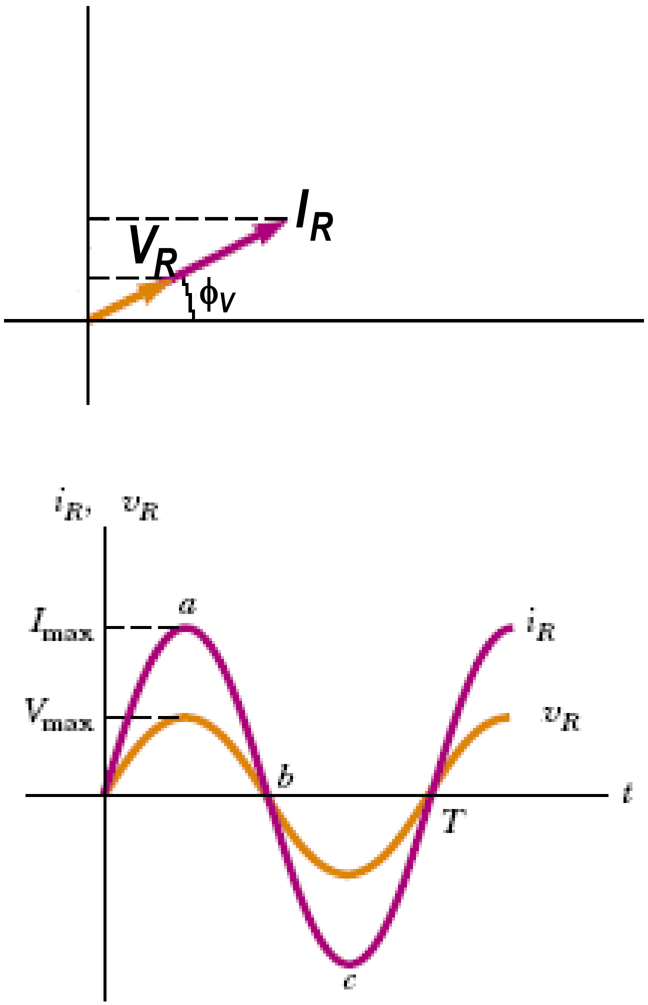
Impedance and Admittance Summary

in the phasor-domain : $V \equiv Z\ I$ and $I \equiv Y\ V$, where:

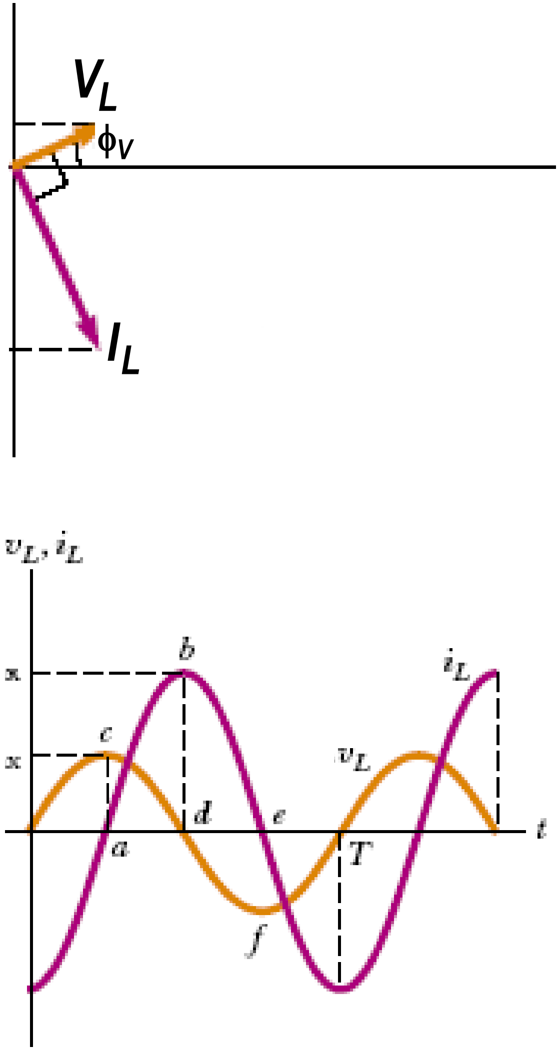
$$Z \equiv \text{impedance } (\Omega) = \begin{cases} R, & \text{for resistors} \\ j\omega L, & \text{for inductors} \\ 1/j\omega C, & \text{for capacitors} \end{cases},$$

$$Y \equiv \text{admittance } \left(1/\Omega\right) = \frac{1}{Z} = \begin{cases} 1/R, & \text{for resistors} \\ 1/j\omega L, & \text{for inductors} \\ j\omega C, & \text{for capacitors} \end{cases}$$

Resistor



Inductor



Capacitor