Note Title 4/17/2007

find the capacitance using the Q-method and the V-method

Solution: Using the Q-method, we start by assuming a surface charge ρ_s on the +ve electrode.

Using boundary conditions

$$D_n = \rho_s \implies \underline{D} = -\rho_s \underline{a}_z$$

$$\implies \underline{E}_1 = \frac{\underline{D}}{\varepsilon_1} = -\frac{\rho_s}{\varepsilon_1}\underline{a}_z \qquad b < z < a$$

also $$\underline{E}_2 = \frac{\underline{D}}{\varepsilon_2} = -\frac{\rho_s}{\varepsilon_2}\underline{a}_z \qquad 0 < z < b$$

We then calculate the voltage difference between the two plate

$$V_a = \int_a^0 \underline{E}\cdot d\underline{l} = \int_a^b \underline{E}_1 \cdot d\underline{l} + \int_b^0 \underline{E}_2 \cdot d\underline{l}$$

$$V_a = V_0 = \int_a^b -\frac{\rho_s}{\varepsilon_1}\underline{a}_z \cdot dz\,\underline{a}_z$$

$$+ \int_b^0 -\frac{\rho_s}{\varepsilon_2}\underline{a}_z \cdot dz\,\underline{a}_z$$

$$\implies V_0 = -\frac{\rho_s}{\varepsilon_1}(b-a) - \frac{\rho_s}{\varepsilon_2} \times -b$$

$$V_0 = \frac{\rho_s}{\varepsilon_1}(a-b) + \frac{\rho_s}{\varepsilon_2} b$$

multiplying both sides by A, the area of the plates, we get

$$V_0 A = \frac{\rho_s A}{\varepsilon_1}(a-b) + \frac{\rho_s A}{\varepsilon_2} b$$

$$V_0 A = Q\left(\frac{(a-b)}{\varepsilon_1} + \frac{b}{\varepsilon_2}\right)$$

$$\Rightarrow C = \frac{Q}{V_0} = \frac{1}{\frac{(a-b)}{\varepsilon_1 A} + \frac{b}{\varepsilon_2 C}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

Two Capacitors in Series!

To use the V-method, we apply Laplace eqn in each region separately because ε is not continuous

In region 1, we have

$$\nabla^2 V_1 = 0 \Longrightarrow \frac{\partial^2 V_1}{\partial z^2} = 0$$

$$V_1 = c_1 z + d_1$$

* Similarly, in region ② we have

$$\nabla^2 V_2 = 0 \Longrightarrow \frac{\partial^2 V_2}{\partial z^2} = 0$$

$$V_2 = c_2 z + d_2$$

* We need 4 eqns to solve for c_1, d_1, c_2, and d_2

* $\underline{V_2(z=0) = 0} \Longrightarrow d_2 = 0 \quad \leftarrow (1)$
* $\underline{V_1(z=a) = V_0} \Longrightarrow c_1 a + d_1 = V_0 \quad \leftarrow (2)$
* $\underline{V_1(z=b) = V_2(z=b)} \Longrightarrow c_1 b + d_1 = c_2 b \quad \leftarrow (3)$
* $\underline{D_{1n}(z=b) = D_{2n}(z=b)} \Longrightarrow \varepsilon_1 c_1 = \varepsilon_2 c_2 \quad \leftarrow (4)$

It follows that we have

$$c_2 = \frac{\varepsilon_1 c_1}{\varepsilon_2} \quad \leftarrow (5)$$

substituting into ③, we get

$$c_1 b + d_1 = \frac{\varepsilon_1 c_1}{\varepsilon_2} b$$

$$\Rightarrow c_1 \left(b - \frac{\varepsilon_1 b}{\varepsilon_2}\right) = -d_1 \Rightarrow d_1 = \left(\frac{\varepsilon_1 b}{\varepsilon_2} - b\right) c_1 \quad (6)$$

Finally, substituting from (6) into (5), we get

$$c_1 a + \left(\frac{\varepsilon_1 b}{\varepsilon_2} - b\right) c_1 = V_0$$

$$\Rightarrow c_1 = \frac{V_0}{a + \left(\frac{\varepsilon_1 b}{\varepsilon_2} - b\right)} = \frac{V_0}{(a-b) + \frac{\varepsilon_1 b}{\varepsilon_2}}$$

* Next step is to get $\underline{E}_1$ from V_1

$$\underline{E}_1 = -\nabla V_1 = -c_1 \underline{a}_z = \frac{-V_0}{(a-b) + \frac{\varepsilon_1 b}{\varepsilon_2}}$$

$$\underline{D}_1 = \varepsilon_1 \underline{E}_1 = \frac{-\varepsilon_1 V_0}{(a-b) + \frac{\varepsilon_1 b}{\varepsilon_2}}$$

* Finally, $\rho_s(z=a) = |D_{1n}| = \dfrac{\varepsilon_1 V_0}{(a-b) + \frac{\varepsilon_1 b}{\varepsilon_2}}$

Multiplying both sides by A, we have

$$\rho_s A = \frac{\varepsilon_1 A V_o}{(a-b) + \frac{\varepsilon_1 b}{\varepsilon_2}}$$

$$Q = \frac{V_o}{\frac{(a-b)}{\varepsilon_1 A} + \frac{b}{\varepsilon_2 A}}$$

$$C = \frac{Q}{V} = \frac{1}{\frac{(a-b)}{\varepsilon_1 A} + \frac{b}{\varepsilon_2 A}}$$

Same answer as Q-method

* repeat the same procedure for Cylindrical & Spherical Coordinates