Lecture 10

From sections 10.1 – 10.2

Solve 10.1

Wave Equation

* We assume first that the electric field has only one component and it is a function only of $z \Rightarrow \underline{E} = E_x \, \underline{a}_x$
* Using Maxwell's equation

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} = -\mu \frac{\partial \underline{H}}{\partial t}$$

$$\Rightarrow -\mu \frac{\partial \underline{H}}{\partial t} = \frac{\partial E_x}{\partial z} \underline{a}_y \Rightarrow \underline{H} = H_y \, \underline{a}_y$$

Wave Equation (Cont'd)

* Also $\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} \Rightarrow$ For a lossless medium with no sources we have

$$\nabla \times \underline{H} = \frac{\partial \underline{D}}{\partial t} \Rightarrow -\frac{\partial H_y}{\partial z}\underline{a}_x = \varepsilon \frac{\partial \underline{E}}{\partial t} = \varepsilon \frac{\partial E_x}{\partial t}\underline{a}_x$$

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \quad — (1) \qquad \varepsilon \frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial z} \quad — (2)$$

2 first order PDEs in two variables that can be reduced to one second order equation in one variable!

Wave Equation (Cont'd)

Differentiate (1) relative to z and (2) relative to t and eliminate $\frac{\partial^2 H_y}{\partial z \partial t}$ to get

$$\frac{\partial^2 E_x}{\partial z^2} = \mu \varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

Define $v_p = \frac{1}{\sqrt{\mu \varepsilon}}$ as the phase velocity

wave eqn: $\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{v_p^2} \frac{\partial^2 E_x}{\partial t^2} = 0$

Solution of wave equation

The solution of this equation is

$$E_x = f_1\left(t - \frac{z}{v_p}\right) + f_2\left(t + \frac{z}{v_p}\right)$$

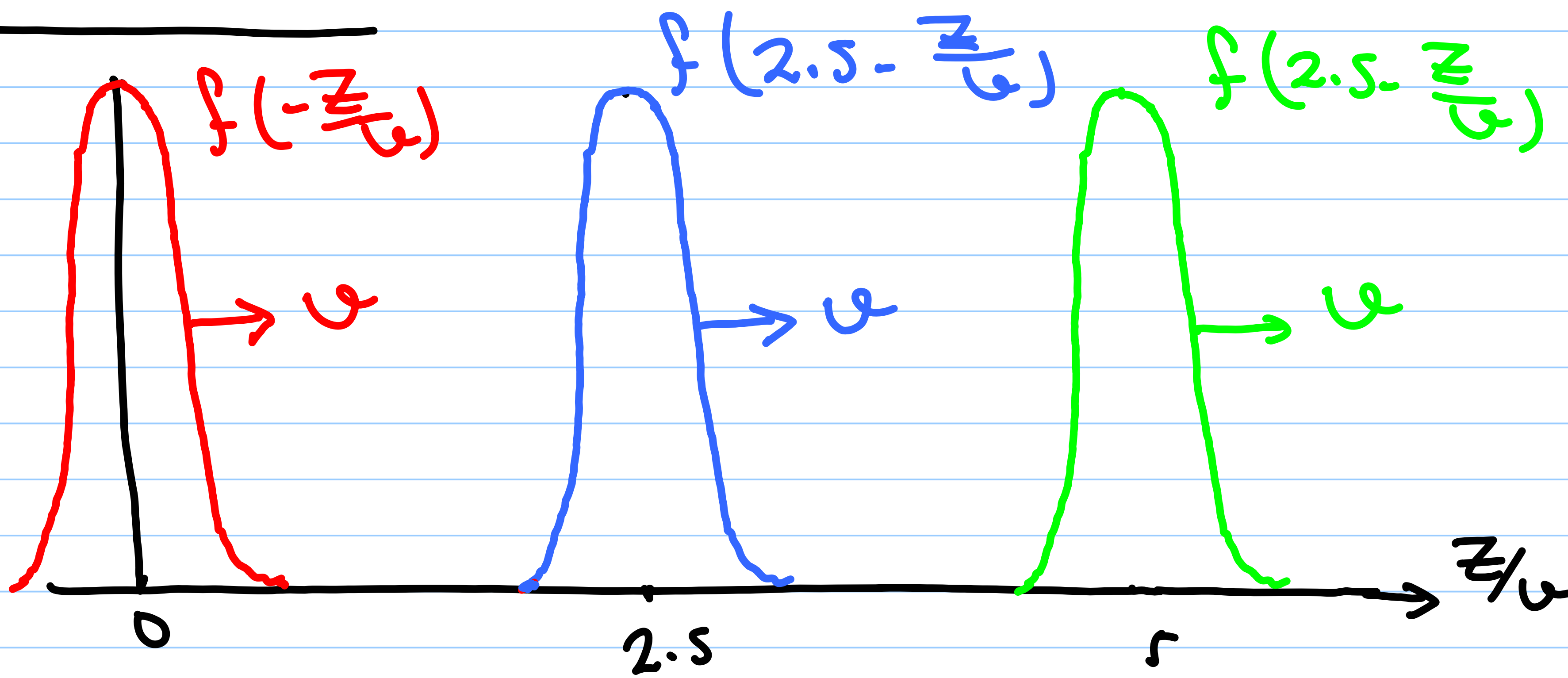
wave travelling in +z direction

wave travelling in −z direction

examples: $f_1\left(t - \frac{z}{v_p}\right) = \cos\left(t - \frac{z}{v_p}\right)$

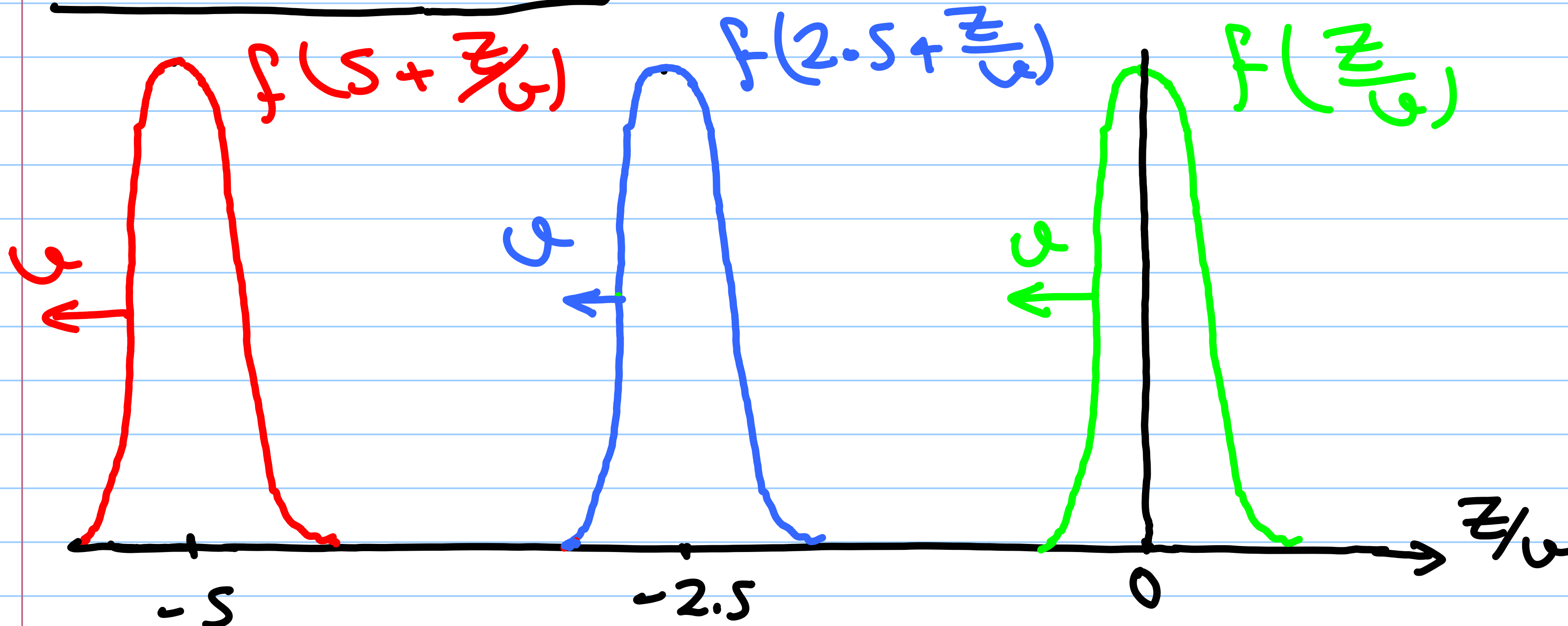
$$f_1\left(t - \frac{z}{v_p}\right) = e^{-\left(t - \frac{z}{v_p}\right)}$$

1D Wave



$$f(\xi) = \exp(-16\xi^2), \quad \xi = t - \frac{z}{v}$$

1D Wave (cont'd)



$$f(\xi) = \exp(-16\xi^2), \quad \xi = t + \frac{z}{v}$$

The Sinsoidal Case

$$f(\xi) = \cos(\omega\xi + \phi)$$

$$\Rightarrow f\left(t - \frac{z}{v}\right) = \cos\left(\omega\left(t - \frac{z}{v}\right) + \phi\right)$$

$$\Rightarrow f\left(t - \frac{z}{v}\right) = \cos\left(\omega t - \frac{\omega}{v} z + \phi\right)$$

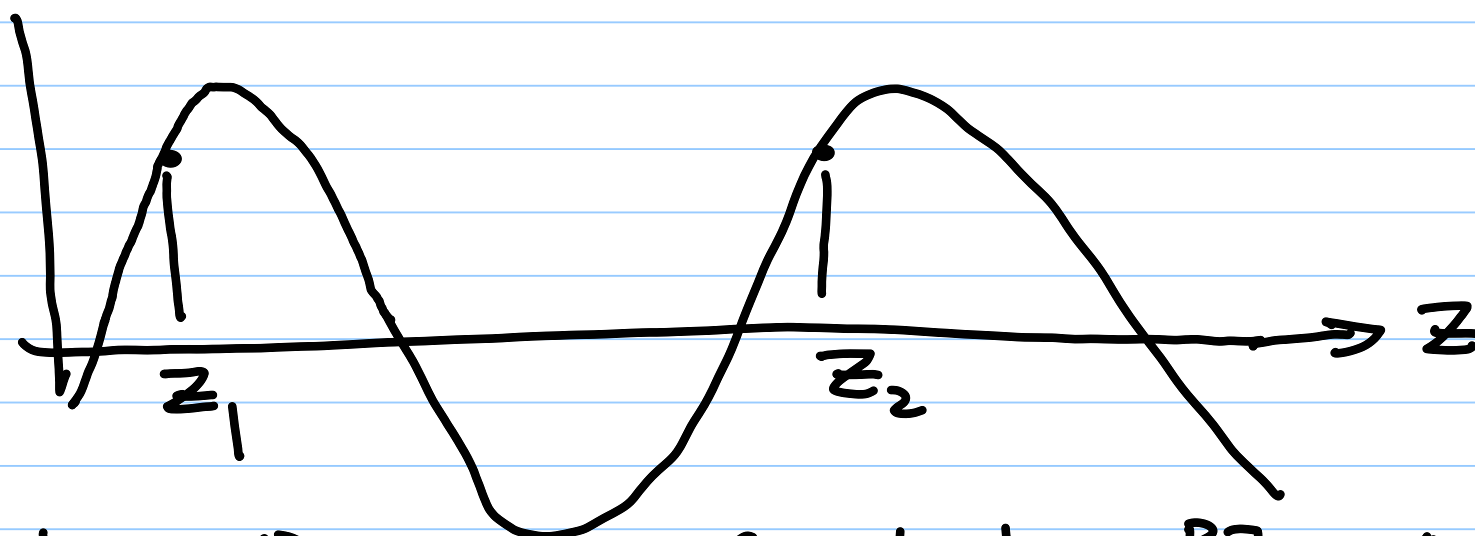
$$\Rightarrow f\left(t - \frac{z}{v}\right) = \cos(\omega t - \beta z + \phi)$$

Similarly, $f\left(t + \frac{z}{v}\right) = \cos(\omega t + \beta z + \phi)$

The sinusoidal Case (Cont'd)

β = propagation Constant $= \frac{\omega}{v}$

* If we fix time, we have



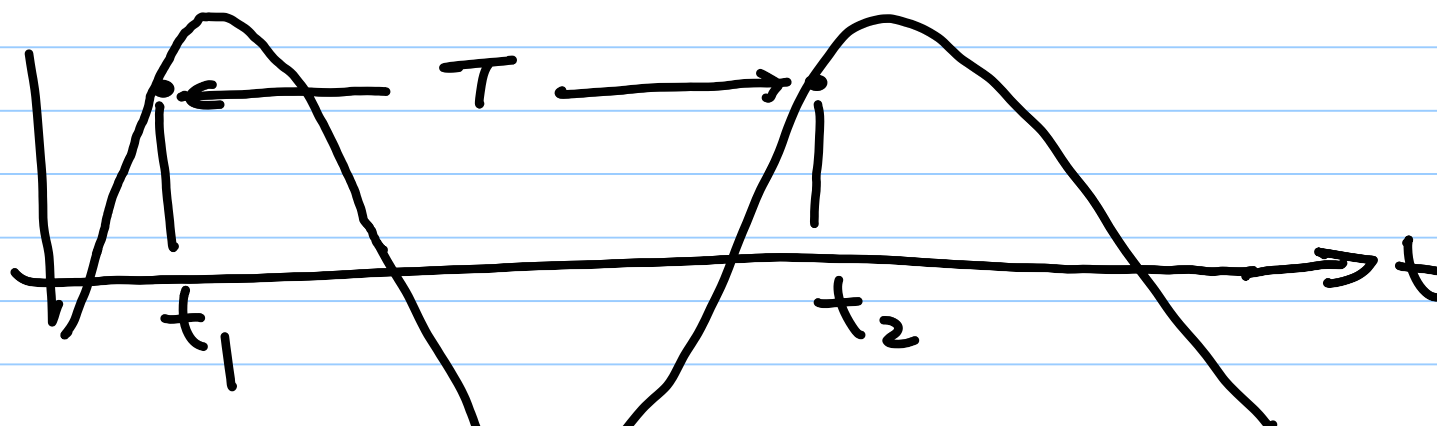
$$\cos(\omega t_0 + \beta z_1 + \phi) = \cos(\omega t_0 + \beta z_2 + \phi)$$

The Sinusoidal Case (Cont'd)

$$\Rightarrow \beta \underbrace{(z_2 - z_1)}_{\lambda} = 2\pi$$

$$\beta\lambda = 2\pi, \quad \lambda = \frac{2\pi}{\beta} = \text{wavelength}$$

* If we fix z, we have



$$\cos(\omega t_1 - \beta z_0 + \phi) = \cos(\omega t_2 - \beta z_0 + \phi)$$

The Sinusoidal Case (Cont'd)

$$\Rightarrow \omega(\underbrace{t_2 - t_1}_{T}) = 2\pi$$

$$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega} = \frac{1}{f}$$

T = periodic time