Lecture 11

From sections 10.3 of Textbook

Solve 10.2 - 10.10

Wave Equation Using Harmonic Fields

$$\nabla \times \underline{\tilde{E}} = -j\omega\mu \underline{\tilde{H}}, \quad \nabla \times \underline{\tilde{H}} = j\omega\varepsilon \underline{\tilde{E}}$$

Take the $\nabla \times$ of the first equation to have $\nabla \times \nabla \times \underline{\tilde{E}} = -j\omega\mu \underbrace{\nabla \times \underline{\tilde{H}}}_{j\omega\varepsilon \underline{\tilde{E}}}$

$$\Rightarrow \nabla \times \nabla \times \underline{\tilde{E}} = \omega^2\mu\varepsilon \underline{\tilde{E}}$$

$$\nabla(\nabla \cdot \underline{\tilde{E}}) - \nabla^2 \underline{\tilde{E}} = \omega^2\mu\varepsilon \underline{\tilde{E}}$$

$\Rightarrow$ For no sources $\nabla \cdot \underline{\tilde{E}} = 0$

Wave Equation (Cont'd)

$\Rightarrow \nabla^2 \tilde{\underline{E}} + \beta^2 \tilde{\underline{E}} = 0$, $\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v_p}$

for x component $\nabla^2 \tilde{E}_x + \beta^2 \tilde{E}_x = 0$

$\Rightarrow \frac{\partial^2 \tilde{E}_x}{\partial x^2} + \frac{\partial^2 \tilde{E}_x}{\partial y^2} + \frac{\partial^2 \tilde{E}_x}{\partial z^2} + \beta^2 \tilde{E}_x = 0$

* Now if the field has only an x component and it is only a function of z, we have $\frac{\partial^2 \tilde{E}_x}{\partial z^2} + \beta^2 \tilde{E}_x = 0$

Wave Equation (Cont'd)

$$\tilde{E}_x = E^+_{x_0} e^{-j\beta z} + E^-_{x_0} e^{j\beta z}$$

+ve z wave   -ve z wave

* Once $\underline{\tilde{E}}$ is known, $\underline{\tilde{H}}$ can be obtained using Maxwell's equations

$$\nabla \times \underline{\tilde{E}} = -j\omega\mu \underline{\tilde{H}}$$

$$\left(\frac{\partial \tilde{E}_x}{\partial z}\right) \underline{a}_y = -j\omega\mu \underline{\tilde{H}} = -j\omega\mu \tilde{H}_y \underline{a}_y$$

$$\tilde{H}_y = -\frac{1}{j\omega\mu}\left(-j\beta E^+_{x_0} e^{-j\beta z} + j\beta E^-_{x_0} e^{j\beta z}\right)$$

Wave Equation (Cont'd)

$$\tilde{H}_y = \frac{\beta}{\omega\mu} E_{x_0}^{+} e^{-jkz} - \frac{\beta}{\omega\mu} E_{x_0}^{-} e^{j\beta z}$$

$$\tilde{H}_y = H_{y_0}^{+} e^{-jkz} + H_{y_0}^{-} e^{jkz}$$

$$\frac{E_{x_0}^{+}}{H_{y_0}^{+}} = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{\left(\frac{\omega}{v_p}\right)} = \mu * \frac{1}{\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}\ \Omega$$

also $$-\frac{E_{x_0}^{-}}{H_{y_0}^{-}} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\varepsilon}}$$

for air $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$

Time Domain Expression

$$\tilde{E}_x(z) = E_{x_0}^{+} e^{-j\beta z} + E_{x_0}^{-} e^{j\beta z}$$

$$\tilde{E}_x(z) = |E_{x_0}^{+}| e^{-j\beta z + j\phi^{+}} + |E_{x_0}^{-}| e^{j\beta z + j\phi^{-}}$$

$$E_x(z,t) = |E_{x_0}^{+}| \cos(\omega t - \beta z + \phi^{+}) + |E_{x_0}^{-}| \cos(\omega t + \beta z + \phi^{-})$$

+z wave

-z wave

propagation in lossy media

* This Derivation is much easier to do in the frequency domain

$$\nabla \times \underline{\tilde{E}} = -j\omega\mu \underline{\tilde{H}}, \quad \nabla \times \underline{\tilde{H}} = \sigma \underline{\tilde{E}} + j\omega\varepsilon \underline{\tilde{E}}$$

$$\Rightarrow \nabla \times \underline{\tilde{H}} = j\omega\varepsilon \left(1 - j\frac{\sigma}{\omega\varepsilon}\right) \underline{\tilde{E}}$$

$$\nabla \times \underline{\tilde{H}} = j\omega \left(\varepsilon - j\frac{\sigma}{\omega}\right) \underline{\tilde{E}} = j\omega(\varepsilon' - j\varepsilon'') \underline{\tilde{E}}$$

* Losses can be modelled using complex permittivity!

Propagation in Lossy Media (Cont'd)

$$K = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon''}$$

$$K = \omega\sqrt{\mu\varepsilon'}\sqrt{1 - j\frac{\varepsilon'}{\varepsilon''}}$$

So, we can obtain

$$\alpha = \mathrm{Re}(jK) = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left(\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right)^{1/2}$$

$$\beta = \mathrm{Im}(jK) = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left(\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right)^{1/2}$$

Propagation in a Lossy Medium (Cont'd)

$$\tilde{\underline{E}} = E_x^+ e^{-jkz} = E_x^+ e^{-(\alpha+j\beta)z} \, a_x$$

$$\underline{E}(z,t) = \mathrm{Re}\left(E_x^+ e^{-(\alpha+j\beta)z} e^{j\omega t} \right) a_x$$

$$\underline{E}(z,t) = \mathrm{Re}\left(|E_x^+| e^{-\alpha z} e^{j(\omega t - \beta z + \phi^+)} \right) a_x$$

$$\underline{E}(z,t) = |E_x^+| e^{-\alpha z} \cos(\omega t - \beta z + \phi^+)$$

Wave decays as it travel in the z direction!

Lossy Medium