TEM Waves and Transmission Lines: Theory

1. TEM Waves

The TEM (Transverse Electro-Magnetic) waves represent a special class of guided EM waves, which is very important from theoretical and practical point of view. They have neither an $\vec{E}$-field component nor an $\vec{H}$-field component in the direction of propagation. For example, if the wave propagates along the *z*-axis, then $E_z = H_z = 0$. There are three important features of the TEM waves that follow from the assumption of vanishing longitudinal field components.

1) The guide propagation constant γ is equal to the free-space propagation constant of the respective medium:

$$\gamma = j\omega\sqrt{\tilde{\mu}\tilde{\varepsilon}} = j\omega\sqrt{(\mu' - j\mu'')[\varepsilon' - j(\varepsilon'' + \sigma/\omega)]}\,. \tag{1}$$

Note that the permeability and the permittivity may be complex numbers due to the presence of magnetic or electric losses.

2) The TEM wave field vectors satisfy the 2-D Laplace equation:

$$\nabla_\perp^2 \vec{a}_\perp = 0, \quad \vec{a}_\perp \equiv \vec{E}_\perp \text{ or } \vec{H}_\perp\,. \tag{2}$$

The differential operator $\nabla_\perp^2$ denotes the Laplace operator in the plane orthogonal to the direction of propagation. In rectangular coordinates, assuming that *z* is the axis of propagation, this operator is

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,. \tag{3}$$

This is a very important conclusion. It implies that:

- The TEM wave $\vec{E}$ field has a distribution in the transverse plane, which does not depend on frequency and is identical with the distribution of the electrostatic field in the structure.
- The TEM wave $\vec{H}$ field has a distribution in the transverse plane, which does not depend on frequency and is identical with the distribution of the magnetostatic field in the structure.

3) Maxwell's equations are simplified to the following form

$$\left|\begin{aligned} &\frac{d}{dz}(\vec{H}_\perp \times \hat{z}) = -j\omega\tilde{\varepsilon}\vec{E}_\perp \\ &\frac{d}{dz}(\vec{E}_\perp \times \hat{z}) = +j\omega\tilde{\mu}\vec{H}_\perp \end{aligned}\right. \tag{4}$$

or

$$\left|\begin{aligned} &\gamma(\vec{H}_\perp \times \hat{z}) = +j\omega\tilde{\varepsilon}\vec{E}_\perp \\ &\gamma(\vec{E}_\perp \times \hat{z}) = -j\omega\tilde{\mu}\vec{H}_\perp. \end{aligned}\right. \tag{5}$$

Equations (4) and (5) show that the vectors $\vec{E}_\perp$ and $\vec{H}_\perp$ are mutually orthogonal.

Equation (2) determines the field distribution in the transverse (*x*-*y*) plane. The field dependence on *z* is pre-determined by the assumption of guided propagation along *z*. For the phasors of time-harmonic field components, this implies that they are functions of the type:

$$\sim f(x,y)e^{\mp\gamma z} = f(x,y)e^{\mp(\alpha + j\beta)z}\,. \tag{6}$$

Such a phasor corresponds to a time-harmonic wave described by the function

$$\sim f(x,y)e^{\mp\alpha z}\cos(\omega t \mp \beta z)\,. \tag{7}$$

Here, β is the phase constant (measured in rad/m), which is related to the wavelength as

$$\beta = \frac{2\pi}{\lambda}\,. \tag{8}$$

The phase constant shows how the phase of a sinusoidal wave, such as the one in (7), changes as the observation point assumes different z positions. For example, if two points P_1 and P_2 are a distance D_{12} apart, the observed phase difference between their sinusoidal waves will be

$$\Delta\varphi_{12} = \beta D_{12} = 2\pi\frac{D_{12}}{\lambda}\text{, rad.} \tag{9}$$

From (1) it follows that the attenuation constant α and the phase constant β can be expressed via the material constants of the medium,

$$\tilde{\mu} = \mu' - j\mu'' \text{ and } \tilde{\varepsilon} = \varepsilon' - j\underbrace{\left(\varepsilon'' + \frac{\sigma}{\omega}\right)}_{\varepsilon''_\sigma}. \tag{10}$$

This is done by solving the system

$$\left|\begin{array}{l}\alpha^2 - \beta^2 = -\omega^2(\mu'\varepsilon' - \mu''\varepsilon''_\sigma)\\ 2\alpha\beta = \omega^2(\mu'\varepsilon''_\sigma + \mu''\varepsilon').\end{array}\right. \tag{11}$$

Solving (11), we obtain

$$\alpha = \omega\sqrt{\frac{\mu'\varepsilon'}{2}}\cdot\sqrt{-1+\sqrt{1+\left(\frac{\mu''}{\mu'}\right)^2+\left(\frac{\varepsilon''+\sigma/\omega}{\varepsilon'}\right)^2}}\,, \tag{12}$$

$$\beta = \omega\sqrt{\frac{\mu'\varepsilon'}{2}}\cdot\sqrt{1+\sqrt{1+\left(\frac{\mu''}{\mu'}\right)^2+\left(\frac{\varepsilon''+\sigma/\omega}{\varepsilon'}\right)^2}}\,. \tag{13}$$

The phase velocity is

$$v_p = \frac{\omega}{\beta} \simeq \left[\sqrt{\frac{\mu'\varepsilon'}{2}}\cdot\sqrt{1+\sqrt{1+\left(\frac{\mu''}{\mu'}\right)^2+\left(\frac{\varepsilon''+\sigma/\omega}{\varepsilon'}\right)^2}}\right]^{-1}. \tag{14}$$

It is obvious from (14) that in lossy media the velocity is frequency dependent, i.e. lossy media are intrinsically dispersive.

Further simplifications follow if (μ'/μ'') and $(\varepsilon'/\varepsilon'')$ are sufficiently small:

$$\alpha \approx \frac{1}{2}\omega\sqrt{\mu'\varepsilon'}\sqrt{\left(\frac{\mu''}{\mu'}\right)^2+\left(\frac{\varepsilon''+\sigma/\omega}{\varepsilon'}\right)^2}\,, \tag{15}$$

$$\beta \approx \omega\sqrt{\mu'\varepsilon'}\,, \tag{16}$$

$$v_p \approx (\mu'\varepsilon')^{-1/2}. \tag{17}$$

In the loss-free case, the expressions (16) and (17) are exact and the phase velocity does not depend on the frequency. Often, there are no magnetic losses in problems of TEM wave propagation, $\mu'' = 0$, as well as the dielectric polarization losses are negligible, $\varepsilon'' \approx 0$. That is why the following approximation of the attenuation constant is widely used:

$$\alpha = \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}} \cdot \sigma . \quad (18)$$

For a *transient* TEM wave, it follows that the field components satisfy the Laplace equation (2) in the transverse plane. With respect to the z variable, they satisfy the time-dependent equation

$$\frac{\partial^2 f}{\partial z^2} - \mu\varepsilon \frac{\partial^2 f}{\partial t^2} - \mu\sigma \frac{\partial f}{\partial t} = 0 , \quad (19)$$

where σ denotes the specific conductivity of the medium (in S/m). When the loss term is weak, $\sigma \ll \omega\varepsilon$ for a given spectral component, the above equation has its general solution in the form

$$\sim f(x, y) e^{\pm \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}\sigma \cdot z} \cdot p(t \pm \frac{z}{v}) . \quad (20)$$

Here, $v = 1/\sqrt{\mu\varepsilon}$ is the approximate velocity of propagation. It becomes the exact velocity in the loss-free case. The traveling wave pulse $p(t \pm z / v)$ can be of any shape. The factor

$$\alpha = \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}} \cdot \sigma \quad (21)$$

in the exponent is the attenuation constant in the case of weakly lossy media. These propagation characteristics were already derived in the case of the time-harmonic waves.

Another characteristic of a TEM wave is its wave impedance. This is also the intrinsic impedance of the medium of propagation. It is defined as the ratio

$$Z_w = \frac{\vec{E}_\perp}{\vec{H}_\perp \times \hat{z}} = \sqrt{\frac{\tilde{\mu}}{\tilde{\varepsilon}}} . \quad (22)$$

The result in (22) follows from (5). In (22), the field vectors must belong to a *traveling* TEM wave. The wave impedance of a TEM wave is a real number in the loss-free case. The wave impedance of a TEM wave in vacuum is $Z_{w0} = \sqrt{\mu_0 / \varepsilon_0} \approx 377 \ \ \Omega$.

2. TEM Transmission Lines

The theory of transmission lines is very similar to that of the TEM waves. Transmission lines, which can support TEM waves (TEM transmission lines), can be analyzed by solving the respective static 2-D problem in their cross-section as dictated by equation (2). Using the electrostatic solution for the $\vec{E}$ field, the capacitance per unit length C_1 is found. The magnetostatic solution provides the value of the inductance per unit length L_1. The theory of stationary (DC) currents in conducting media allows us to determine the resistance per unit length R_1 and the conductance per unit length G_1 of the transmission line. Thus, a very short portion of a transmission line of length Δz can be represented by the equivalent circuit shown in Fig. 1.

Applying Kirchhoff's laws to the circuit in Fig. 1, we obtain the telegrapher's equations:

$$\left| \begin{aligned} \frac{dV}{dz} &= -(R_1 + j\omega L_1) I \\ \frac{dI}{dz} &= -(G_1 + j\omega C_1) V . \end{aligned} \right. \quad (23)$$

The system of equations (23) can be reduced to a single second-order equation for either V or I:

$$\frac{d^2 V}{dz^2} - \gamma^2 V = 0 . \quad (24)$$

Here,

$$\gamma = \sqrt{(R_1 + j\omega L_1)(G_1 + j\omega C_1)} \quad (25)$$

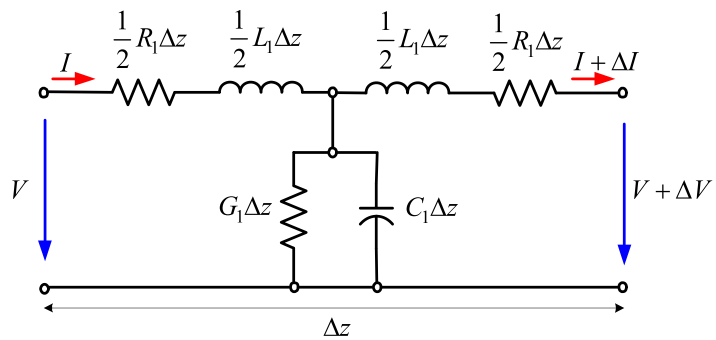

Fig. 1. The equivalent circuit of a transmission-line portion of length Δz.

is the transmission line's propagation constant. The analogy between the propagation constant of a TEM wave in (1) and the transmission line's propagation constant becomes obvious if one transforms (25) to the following equivalent expression:

$$\gamma = \alpha + j\beta = j\omega\sqrt{\left(L_1 - j\frac{R_1}{\omega}\right)\left(C_1 - j\frac{G_1}{\omega}\right)}. \tag{26}$$

Using this analogy, one can readily express the attenuation constant α and the phase constant β in terms of L_1, C_1, R_1 and G_1 as

$$\alpha = \omega\sqrt{\frac{L_1C_1}{2}}\sqrt{-1+\sqrt{1+\left(\frac{G_1}{\omega C}\right)^2+\left(\frac{R_1}{\omega L}\right)^2}}, \tag{27}$$

$$\beta = \omega\sqrt{\frac{L_1C_1}{2}}\sqrt{1+\sqrt{1+\left(\frac{G_1}{\omega C}\right)^2+\left(\frac{R_1}{\omega L}\right)^2}}. \tag{28}$$

The voltage and current 1-D waves propagate along the line with velocity

$$v = \frac{\omega}{\beta} \simeq \left[\sqrt{\frac{L_1C_1}{2}}\sqrt{1+\sqrt{1+\left(\frac{G_1}{\omega C}\right)^2+\left(\frac{R_1}{\omega L}\right)^2}}\right]^{-1}. \tag{29}$$

When the loss is negligible,

$$v = (L_1C_1)^{-1/2}, \tag{30}$$

$$\beta = \omega\sqrt{L_1C_1}, \tag{31}$$

and

$$\alpha \approx \frac{1}{2}\omega\sqrt{L_1C_1}\sqrt{\left(\frac{R_1}{\omega L_1}\right)^2+\left(\frac{G_1}{\omega C_1}\right)^2}. \tag{32}$$

In transmission lines, it is common that conduction leakage is practically eliminated, $G_1 \approx 0$. However, resistive losses in the guiding wires cannot be avoided. That is why, the following approximation of the attenuation constant is often used in practice:

$$\alpha \approx \frac{1}{2}\frac{R_1}{\sqrt{L_1/C_1}}. \tag{33}$$

The characteristic impedance of a TEM line is essential when matching is required. It is defined as the ratio

$$Z_c = V/I \tag{34}$$

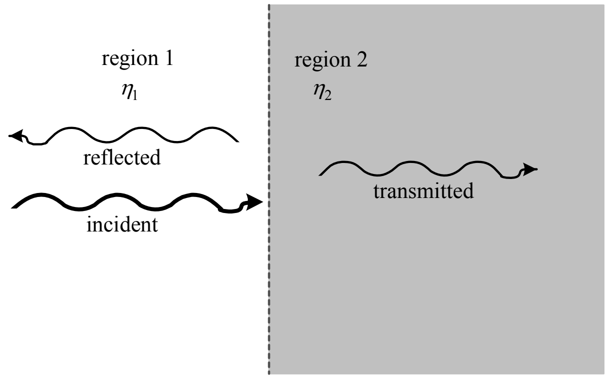

Fig. 2. Normal incidence from region 1.

for a traveling wave. It is clear from (23) that

$$Z_c = \sqrt{\frac{R_1 + j\omega L_1}{G_1 + j\omega C_1}} . \tag{35}$$

In a loss-free transmission line,

$$Z_c = \sqrt{L_1 / C_1} . \tag{36}$$

From (29) it is obvious that real transmission lines, which always have some (albeit small) loss, exhibit some dispersion. The dispersion in transmission lines is defined as the velocity of propagation dependence on the frequency of the signal. This means that a lossy transmission line of significant length distorts a non-sinusoidal signal (e.g., a pulse) because the different spectral components of this signal experience different delays. The velocity of propagation of a pulse is better estimated by the *group velocity*, which is defined by the relation

$$v_g = \left(\frac{d\beta}{d\omega} \right)^{-1} . \tag{37}$$

This is the velocity of propagation of the signal's envelope. Obviously, the group velocity is equal to the phase velocity in dispersion-free media.

3. Reflection and Transmission of TEM Waves at Discontinuities

Here, only normal incidence is considered. The reflection coefficient Γ is defined as the ratio of the reflected and the incident field at the discontinuity:

$$\Gamma = \frac{E_0^r}{E_0^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} , \tag{38}$$

where η_1 and η_2 denote the intrinsic impedance of the two regions: region 1 is the place of origin of the incident wave, and region 2 is where the transmitted wave propagates. The transmission coefficient is the ratio of the transmitted and the incident field at the discontinuity:

$$T = \frac{E_0^t}{E_0^i} = \frac{2\eta_2}{\eta_2 + \eta_1} . \tag{39}$$

The continuity of the tangential field components at the interface requires that

$$T = 1 + \Gamma . \tag{40}$$

4. Standing Wave Ratio (SWR) in Transmission Lines

The general solution of the 1-D wave equation (24) consists of two terms: a wave propagating along $+z$ and a wave propagating along $-z$. The phasor of the $+z$ wave (the incident wave) is

$$V^i(z) = V_m^i e^{-\gamma z} = V_m^i e^{-\alpha z} e^{-j\beta z} . \tag{41}$$

Its time-dependent counterpart is

$$v^i(z,t) = V_m^i e^{-\alpha z} \cos(\omega t - \beta z) . \tag{42}$$

The respective expressions for the $-z$ wave (the reflected wave) are:

$$V^r(z) = V_m^r e^{+\gamma z} = V_m^r e^{+\alpha z} e^{+j(\beta z + \Delta\varphi)} , \tag{43}$$

$$v^r(z,t) = V_m^r e^{+\alpha z} \cos(\omega t + \beta z + \Delta\varphi) . \tag{44}$$

In a TEM transmission line, reflection occurs any time when there is a discontinuity of the characteristic impedance of the line. The reflection coefficient is defined as in (38) only that one has to replace the intrinsic wave impedances with the characteristic impedances:

$$\Gamma = \frac{Z_{c_2} - Z_{c_1}}{Z_{c_2} + Z_{c_1}} . \tag{45}$$

Same holds for the transmission coefficient:

$$T = \frac{2Z_{c_2}}{Z_{c_2} + Z_{c_1}} = 1 + \Gamma . \tag{46}$$

When a wave propagates along an infinite transmission line, or a transmission line, which is perfectly matched to its load ($\Gamma = 0$), there is no reflected wave. A single *traveling wave* is present. The measured RMS value of the voltage (or the current) is the same all along the line.

At the other extreme is the case of total reflection, $|\Gamma| = 1$. There are two traveling waves propagating in opposite directions: the incident wave and the reflected wave, which have the same magnitudes. It can be shown that there are points along the line where the measured RMS value of the voltage (the current) is zero. These are the points where both waves interfere destructively. They are called the *nulls* of the wave. At other points (a quarter-wavelength away from the nulls), the maxima of the wave can be measured. These maxima are twice the magnitude of the incident wave. This is the case of a pure *standing wave*, when there is no energy transfer along the line.

A simple example is a short-circuited transmission line. There, $\Gamma = -1$ and the total wave in the line is

$$v(z,t) = V_m[\underbrace{\cos(\omega t - \beta z)}_{\text{incident}} - \underbrace{\cos(\omega t + \beta z)}_{\text{reflected}}] = 2V_m \cdot \cos(\omega t) \cdot \sin(\beta z) . \tag{47}$$

In between the above extremes lie infinite number of combinations of incident and reflected waves of different magnitude and phase depending on the reflection coefficient Γ. However, they all can be represented as a superposition of a traveling wave and a standing wave. Where the standing wave and the traveling wave interfere constructively, the maxima of the total wave occur. These maxima are $(1+|\Gamma|)$ times the magnitude of the incident wave. Where the standing wave and the traveling wave interfere destructively, the minima of the total wave occur, which are $(1-|\Gamma|)$ times the magnitude of the incident wave. In order to describe the matching conditions of the transmission line, the *SWR* (Standing Wave Ratio) is introduced as the ratio of the maxima and the minima of the total wave in the transmission line:

$$SWR = \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|} . \tag{48}$$

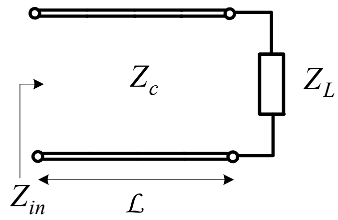

Fig. 3. One-port network consisting of a loss-free transmission line and a load.

The *SWR* has a minimum value of 1 in the case of a traveling wave. It is infinity in the case of a standing wave. In RF and microwave engineering, a $SWR \le 2$ is considered satisfactory for the purposes of matching and good power transfer. At $SWR = 2$, about 90% of the incident power is transferred to the load and about 10% is reflected back to the generator.

5. Impedance Matching

Impedance matching is a major concern in the design of power transmission systems and high-frequency electronic circuits and waveguides. Impedance matching ensures that all electromagnetic energy is being transferred to the load, i.e., there is no reflection. The perfect match is achieved when $\Gamma = 0$ (or *SWR* = 1). When a load is connected to a transmission line, the reflection coefficient at the load Γ_L is calculated as

$$\Gamma_L = \frac{Z_L - Z_c}{Z_L + Z_c} = Z_c \frac{z_L - 1}{z_L + 1}. \tag{49}$$

Here, $z_L = Z_L / Z_c$ is the normalized load impedance, and Z_c is the characteristic impedance of the line. We will consider only lossless lines. They have characteristic impedance, which is a real number, $Z_c = \sqrt{\mu / \varepsilon}$.

Transmission lines have one very useful feature: they act as impedance transformers. A distance $\mathcal{L}$ from the source, the input impedance of the 1-port network (see Fig. 3) becomes:

$$Z_{in} = Z_c \frac{Z_L + jZ_c \tan \beta\mathcal{L}}{Z_c + jZ_L \tan \beta\mathcal{L}}, \tag{50}$$

where $\beta = 2\pi / \lambda$ is the phase constant.

One classical example of a simple impedance transformer is the quarter-wavelength transformer. Assume that a load Z_L has to be transformed to a certain input impedance Z_{in}. One can readily obtain an expression for the line impedance Z_c if the length of the transmission line is set to $\mathcal{L} = \lambda / 4$ from (50):

$$Z_c = \sqrt{Z_{in} Z_L}. \tag{51}$$

Thus, a transmission line of length $\mathcal{L} = \lambda / 4$ and characteristic impedance defined by (51) will match the load Z_L to a source impedance of $Z_s = Z_{in}^*$ perfectly[1]. Of course, the impedance matching properties of this single-section transformer are very narrow-band because they depend on the wavelength $\lambda = v_p / f$.

6. The Virtual TEM Laboratory

In the TEM Wave and Transmission Lines Laboratory Module, a transient 2-D electromagnetic simulator will be used: MEFiSTo-2D Classic from Faustus Scientific

[1] From circuit theory, it is known that the maximum power transfer occurs if the load impedance equals the complex conjugate of the source impedance.

Corporation.[2] MEFiSTo-2D is a time-domain solver. It solves the electromagnetic partial differential equations directly in time and in space. It has an excellent post-processor, which allows the visualization of the EM wave propagation. The user can watch simultaneous display of the time-response, its envelope, its amplitude spectrum and its phase spectrum.

In all examples, we will use parallel-plate lines which consist of two parallel metallic strips. The graphic user interface (GUI) of MEFiSTo-2D is a 2-D viewer where the strip is clearly indicated by blue contour lines. The field in such lines can be approximately described by a TEM plane wave whose $\vec{E}$ field points from one plate to the other while the $\vec{H}$ field is parallel to the plate and orthogonal to the length of the strips (i.e., the direction of propagation). The $\vec{E}$ and $\vec{H}$ vectors are constant in the cross-section of the line, which means that they satisfy the condition in equation (2) for a TEM wave. When the field is constant in a plane orthogonal to the direction of propagation, it is called a *uniform plane wave*, which is a special case of the TEM wave.

Using the formula for the capacitance of a parallel-plate capacitor, it is easy to find the capacitance per unit length of a parallel-plate line as

$$C_1 = \varepsilon \frac{w}{h}, \text{ F/m} \tag{52}$$

where w is the width of the strips, h is the distance between them, and ε is the permittivity of the medium sandwiched between the plates. On the other hand, we know that the phase velocity of the plane wave in a TEM line is the same as that in open space; therefore,

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{L_1 C_1}}. \tag{53}$$

From (52) and (53), we find that

$$L_1 = \mu \frac{h}{w}, \text{ H/m.} \tag{54}$$

Now we can determine the characteristic impedance of a parallel-plate line as

$$Z_c = \sqrt{\frac{L_1}{C_1}} = \sqrt{\frac{\mu}{\varepsilon}} \cdot \frac{h}{w} = Z_i \cdot \frac{h}{w}, \ \Omega \tag{55}$$

where Z_i is the intrinsic impedance of the medium sandwiched between the two parallel strips of the line.

[2] MEFiSTo-2D™ Classic is a student freeware available for download from https://www.faustcorp.com/cgi-bin/downloads/classic/classic_request/classic_index.cgi.