REVIEW QUESTIONS

2.1 The ranges of θ and ϕ as given by eq. (2.17) are not the only possible ones. The following are all alternative ranges of θ and ϕ, except

(a) $0 \le \theta < 2\pi, 0 \le \phi \le \pi$
(b) $0 \le \theta < 2\pi, 0 \le \phi < 2\pi$
(c) $-\pi \le \theta \le \pi, 0 \le \phi \le \pi$
(d) $-\pi/2 \le \theta \le \pi/2, 0 \le \phi < 2\pi$
(e) $0 \le \theta \le \pi, -\pi \le \phi < \pi$
(f) $-\pi \le \theta < \pi, -\pi \le \phi < \pi$

2.2 At Cartesian point $(-3, 4, -1)$, which of these is incorrect?

(a) $\rho = -5$
(b) $r = \sqrt{26}$
(c) $\theta = \tan^{-1}\dfrac{5}{-1}$
(d) $\phi = \tan^{-1}\dfrac{4}{-3}$

2.3 Which of these is not valid at point $(0, 4, 0)$?

(a) $\mathbf{a}_\phi = -\mathbf{a}_x$
(b) $\mathbf{a}_\theta = -\mathbf{a}_z$
(c) $\mathbf{a}_r = 4\mathbf{a}_y$
(d) $\mathbf{a}_\rho = \mathbf{a}_y$

2.4 A unit normal vector to the cone $\theta = 30°$ is:

(a) $\mathbf{a}_r$
(b) $\mathbf{a}_\theta$
(c) $\mathbf{a}_\phi$
(d) none of these

2.5 At every point in space, $\mathbf{a}_\phi \cdot \mathbf{a}_\theta = 1$.

(a) True
(b) False

2.6 If $\mathbf{H} = 4\mathbf{a}_\rho - 3\mathbf{a}_\phi + 5\mathbf{a}_z$, at $(1, \pi/2, 0)$ the component of $\mathbf{H}$ parallel to surface $\rho = 1$ is

(a) $4\mathbf{a}_\rho$
(b) $5\mathbf{a}_z$
(c) $-3\mathbf{a}_\phi$
(d) $-3\mathbf{a}_\phi + 5\mathbf{a}_z$
(e) $5\mathbf{a}_\phi + 3\mathbf{a}_z$

2.7 Given $\mathbf{G} = 20\mathbf{a}_r + 50\mathbf{a}_\theta + 40\mathbf{a}_\phi$, at $(1, \pi/2, \pi/6)$ the component of $\mathbf{G}$ perpendicular to surface $\theta = \pi/2$ is

(a) $20\mathbf{a}_r$
(b) $50\mathbf{a}_\theta$
(c) $40\mathbf{a}_\phi$
(d) $20\mathbf{a}_r + 40\mathbf{a}_\theta$
(e) $-40\mathbf{a}_r + 20\mathbf{a}_\phi$

2.8 Where surfaces $\rho = 2$ and $z = 1$ intersect is

(a) an infinite plane
(b) a semi-infinite plane
(c) a circle
(d) a cylinder
(e) a cone

2.9 Match the items in the list at the left with those in the list at the right. Each answer can be used once, more than once, or not at all.

(a) $\theta = \pi/4$	(i) infinite plane
(b) $\phi = 2\pi/3$	(ii) semi-infinite plane
(c) $x = -10$	(iii) circle
(d) $r = 1, \theta = \pi/3, \phi = \pi/2$	(iv) semicircle
(e) $\rho = 5$	(v) straight line
(f) $\rho = 3, \phi = 5\pi/3$	(vi) cone
(g) $\rho = 10, z = 1$	(vii) cylinder
(h) $r = 4, \phi = \pi/6$	(viii) sphere
(i) $r = 5, \theta = \pi/3$	(ix) cube
	(x) point

2.10 A wedge is described by $z = 0$, $30° < \phi < 60°$. Which of the following is incorrect?

(a) The wedge lies in the $x - y$ plane.
(b) It is infinitely long.
(c) On the wedge, $0 < \rho < \infty$.
(d) A unit normal to the wedge is $\pm\mathbf{a}_z$.
(e) The wedge includes neither the x-axis nor the y-axis.

Answers: 2.1b,f, 2.2a, 2.3c, 2.4b, 2.5b, 2.6d, 2.7b, 2.8c, 2.9a-(vi), b-(ii), c-(i), d-(x), e-(vii), f-(v), g-(iii), h-(iv), i-(iii), 2.10b.

PROBLEMS

Section 2.3 and 2.4—Cylindrical and Spherical Coordinates

2.1 Express the following points in Cartesian coordinates:

(a) $P_1(2, 30°, 5)$
(b) $P_2(1, 90°, -3)$
(c) $P_3(10, \pi/4, \pi/3)$
(d) $P_4(4, 30°, 60°)$

2.2 Express the following points in cylindrical and spherical coordinates:

(a) $P(1, -4, -3)$
(b) $Q(3, 0, 5)$
(c) $R(-2, 6, 0)$

2.3 The rectangular coordinates at point P are $(x = 2, y = 6, z = -4)$. (a) What are its cylindrical coordinates? (b) What are its spherical coordinates?

2.4 The cylindrical coordinates of point Q are $\rho = 5, \phi = 120°, z = 1$. Express Q as rectangular and spherical coordinates.

2.5 A point in spherical coordinates is given by $P(4, \pi/2, \pi/3)$. Express the point in Cartesian and cylindrical coordinates.

2.6 (a) If $V = xz - xy + yz$, express V in cylindrical coordinates.
(b) If $U = x^2 + 2y^2 + 3z^2$, express U in spherical coordinates.

2.7 Convert the following vectors to cylindrical and spherical systems:

(a) $\mathbf{F} = \dfrac{x\mathbf{a}_x + y\mathbf{a}_y + 4\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$

(b) $\mathbf{G} = (x^2 + y^2)\left[\dfrac{x\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} + \dfrac{y\mathbf{a}_y}{\sqrt{x^2 + y^2 + z^2}} + \dfrac{z\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}\right]$

2.8 Let $\mathbf{F} = \dfrac{10(x\mathbf{a}_x + y\mathbf{a}_y)}{x^2 + y^2}$.

(a) Express $\mathbf{F}$ in cylindrical coordinates.
(b) Find $\mathbf{F}$ at point $P(1, \pi/2, 4)$ and express your result in both rectangular and cylindrical coordinates.

2.9 Given vector $\mathbf{A} = 2\mathbf{a}_\rho + 3\mathbf{a}_\phi + 4\mathbf{a}_z$, convert $\mathbf{A}$ into Cartesian coordinates at point $(2, \pi/2, -1)$.

2.10 Express the following vectors in rectangular coordinates:

(a) $\mathbf{A} = \rho \sin\phi\, \mathbf{a}_\rho + \rho\cos\phi\, \mathbf{a}_\phi - 2z\, \mathbf{a}_z$
(b) $\mathbf{B} = 4r\cos\phi\, \mathbf{a}_r + r\, \mathbf{a}_\theta$

2.11 Let $\mathbf{H} = \rho z \sin\phi \mathbf{a}_\rho - \rho(z+1)\cos\phi \mathbf{a}_\phi + \rho^2 z \mathbf{a}_z$. Express $\mathbf{H}$ in rectangular coordinates.

2.12 Given that $\mathbf{F} = yz\mathbf{a}_x + xz\mathbf{a}_y + x^2\mathbf{a}_z$, express $\mathbf{F}$ in cylinderical and spherical coordinates.

2.13 Prove the following:

(a)
$$\mathbf{a}_x \cdot \mathbf{a}_\rho = \cos\phi$$
$$\mathbf{a}_x \cdot \mathbf{a}_\phi = -\sin\phi$$
$$\mathbf{a}_y \cdot \mathbf{a}_\rho = \sin\phi$$
$$\mathbf{a}_y \cdot \mathbf{a}_\phi = \cos\phi$$

(b)
$$\mathbf{a}_x \cdot \mathbf{a}_r = \sin\theta\cos\phi$$
$$\mathbf{a}_x \cdot \mathbf{a}_\theta = \cos\theta\cos\phi$$
$$\mathbf{a}_y \cdot \mathbf{a}_r = \sin\theta\sin\phi$$

(c)
$$\mathbf{a}_y \cdot \mathbf{a}_\theta = \cos\theta\sin\phi$$
$$\mathbf{a}_z \cdot \mathbf{a}_r = \cos\theta$$
$$\mathbf{a}_z \cdot \mathbf{a}_\theta = -\sin\theta$$

2.14 (a) Show that point transformation between cylindrical and spherical coordinates is obtained using

$$r = \sqrt{\rho^2 + z^2}, \quad \theta = \tan^{-1}\frac{\rho}{z}, \quad \phi = \phi$$

or

$$\rho = r\sin\theta, \quad z = r\cos\theta, \quad \phi = \phi$$

(b) Show that vector transformation between cylindrical and spherical coordinates is obtained using

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

or

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

(*Hint:* Make use of Figures 2.5 and 2.6.)

2.15 A vector $\mathbf{F}$ is represented in rectangular coordinates as
$\mathbf{F} = 10\mathbf{a}_x - 4\mathbf{a}_y + 2\mathbf{a}_z$. Find its representation using the following base vectors:

$$\mathbf{a}_1 = \frac{1}{\sqrt{2}}(\mathbf{a}_x - \mathbf{a}_z), \qquad \mathbf{a}_2 = \frac{1}{\sqrt{2}}(\mathbf{a}_x + \mathbf{a}_z), \qquad \mathbf{a}_3 = \mathbf{a}_y$$

2.16 Show that the vector fields

$$\mathbf{A} = \rho\sin\phi\mathbf{a}_\rho + \rho\cos\phi\mathbf{a}_\phi - \rho\mathbf{a}_z$$
$$\mathbf{B} = \rho\sin\phi\mathbf{a}_\rho + \rho\cos\phi\mathbf{a}_\phi - \rho\mathbf{a}_z$$

are perpendicular to each other at any point.

2.17 Given that $\mathbf{A} = 3\mathbf{a}_\rho + 2\mathbf{a}_\phi + \mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_\rho - 8\mathbf{a}_z$, find:

(a) $\mathbf{A} + \mathbf{B}$, (b) $\mathbf{A}\cdot\mathbf{B}$, (c) $\mathbf{A}\times\mathbf{B}$, (d) the angle between $\mathbf{A}$ and $\mathbf{B}$.

2.18 Let $\mathbf{A} = \rho\cos\phi\, \mathbf{a}_\rho + \rho z^2 \sin\phi\, \mathbf{a}_z$.

(a) Transform $\mathbf{A}$ into rectangular coordinates and calculate its magnitude at point $(3, -4, 0)$.
(b) Transform $\mathbf{A}$ into spherical system and calculate its magnitude at point $(3, -4, 0)$.

2.19 The transformation $(A_\rho, A_\phi, A_z) \rightarrow (A_x, A_y, A_z)$ in eq. (2.15) is not complete. Complete it by expressing $\cos\phi$ and $\sin\phi$ in terms of x, y, and z. Do the same thing to the transformation $(A_r, A_\theta, A_\phi) \rightarrow (A_x, A_y, A_z)$ in eq. (2.28).

2.20 In Practice Exercise 2.2, express **A** in spherical and **B** in cylindrical coordinates. Evaluate **A** at $(10, \pi/2, 3\pi/4)$ and **B** at $(2, \pi/6, 1)$.

2.21 Calculate the distance between the following pairs of points:

(a) $(2, 1, 5)$ and $(6, -1, 2)$
(b) $(3, \pi/2, -1)$ and $(5, 3\pi/2, 5)$
(c) $(10, \pi/4, 3\pi/4)$ and $(5, \pi/6, 7\pi/4)$

2.22 Given points $P(10, \pi/4, 0)$ and $Q(4, \pi/2, \pi/2)$, find the distance between P and Q.

2.23 Describe the intersection of the following surfaces:

(a) $x = 2, \quad y = 5$
(b) $x = 2, \quad y = -1, \quad z = 10$
(c) $r = 10, \quad \theta = 30°$
(d) $\rho = 5, \quad \phi = 40°$
(e) $\phi = 60°, z = 10$
(f) $r = 5, \quad \phi = 90°$

2.24 At point $T(2, 3, -4)$, express $\mathbf{a}_z$ in the spherical system and $\mathbf{a}_r$ in the rectangular system.

2.25 Let $\mathbf{A} = (2z - \sin\phi)\mathbf{a}_\rho + (4\rho + 2\cos\phi)\mathbf{a}_\phi - 3\rho z\mathbf{a}_z$ and $\mathbf{B} = \rho\cos\phi\mathbf{a}_\rho + \sin\phi\mathbf{a}_\phi + \mathbf{a}_z$.

(a) Find the minimum angle between **A** and **B** at $(1, 60°, -1)$.
(b) Determine a unit vector normal to both **A** and **B** at $(1, 90°, 0)$.

2.26 Given vectors $\mathbf{A} = 2\mathbf{a}_x + 4\mathbf{a}_y + 10\mathbf{a}_z$ and $\mathbf{B} = -5\mathbf{a}_\rho + \mathbf{a}_\phi - 3\mathbf{a}_z$, find

(a) $\mathbf{A} + \mathbf{B}$ at $P(0, 2, -5)$
(b) The angle between **A** and **B** at P
(c) The scalar component of **A** along **B** at P

2.27 Given that $\mathbf{G} = 6r^2\sin\phi\,\mathbf{a}_r + r^2\mathbf{a}_\phi$, find $\mathbf{G}\cdot\mathbf{a}_y$ at $(2, -3, 1)$.

2.28 A vector field in "mixed" coordinate variables is given by

$$\mathbf{G} = \frac{x\cos\phi}{\rho}\mathbf{a}_x + \frac{2yz}{\rho^2}\mathbf{a}_y + \left(1 - \frac{x^2}{\rho^2}\right)\mathbf{a}_z$$

Express **G** completely in the spherical system.

Section 2.5—Constant-Coordinate Surfaces

2.29 If $\mathbf{J} = r\sin\theta\cos\phi\,\mathbf{a}_r - \cos 2\theta\sin\phi\,\mathbf{a}_\theta + \tan\frac{\theta}{2}\ln r\,\mathbf{a}_\phi$ at $T(2, \pi/2, 3\pi/2)$, determine the vector component of **J** that is:

(a) Parallel to $\mathbf{a}_z$
(b) Normal to surface $\phi = 3\pi/2$
(c) Tangential to the spherical surface $r = 2$
(d) Parallel to the line $y = -2, z = 0$

2.30 Let $\mathbf{H} = 5\rho\sin\phi\,\mathbf{a}_\rho - \rho z\cos\phi\,\mathbf{a}_\phi + 2\rho\,\mathbf{a}_z$. At point $P(2, 30°, -1)$, find:

(a) A unit vector along **H**
(b) The component of **H** parallel to $\mathbf{a}_x$
(c) The component of **H** normal to $\rho = 2$
(d) The component of **H** tangential to $\phi = 30°$

2.31 If $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$, describe the surface defined by:

(a) $\mathbf{r}\cdot\mathbf{a}_x + \mathbf{r}\cdot\mathbf{a}_y = 5$
(b) $|\mathbf{r}\times\mathbf{a}_z| = 10$