2.2 Express these points in Cylindrical & Spherical coordinates?

a) $P = (1, -4, -3)$

$P(\rho, \varphi, z) = ?$

$\rho = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-4)^2} = \sqrt{17} = 4.1231$

$\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-4}{1}\right) = 284.0362^\circ$

$z = z = -3$

$\Rightarrow P(\rho, \varphi, z) = (4.1231, 284.0362^\circ, -3)$

$P(r, \theta, \varphi) = ?$

$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + (-4)^2 + (-3)^2} = \sqrt{26} = 5.0990$

$\theta = \tan^{-1}\left(\frac{\rho}{z}\right) = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{17}}{-3}\right) = 126.0399^\circ$

$\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-4}{1}\right) = 284.0362^\circ$

$\Rightarrow P(r, \theta, \varphi) = (5.0990, 126.0399^\circ, 284.0362^\circ)$

b) $Q = (3, 0, 5)$

$Q(\rho, \varphi, z) = ?$

$\rho = \sqrt{x^2 + y^2} = \sqrt{3^2 + 0^2} = 3$

$\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{3}\right) = 0^\circ$

$z = z = 5$

$\Rightarrow Q(\rho, \varphi, z) = (3, 0^\circ, 5)$

2.2

b)

$Q(r,\theta,\varphi)=?$

$r=\sqrt{x^2+y^2+z^2}=\sqrt{3^2+0^2+5^2}=\sqrt{34}=5.8310$

$\theta=\tan^{-1}\left(\frac{\rho}{z}\right)=\tan^{-1}\left(\frac{3}{5}\right)=30.9638^\circ$

$\varphi=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{0}{3}\right)=0^\circ$

$\Rightarrow Q(r,\theta,\varphi)=(5.8310,\ 30{,}9638^\circ,\ 0^\circ)$

c) $R=(-2,6,0)$

$R(\rho,\varphi,z)=?$

$\rho=\sqrt{x^2+y^2}=\sqrt{(-2)^2+6^2}=\sqrt{40}=6.3246$

$\varphi=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{6}{-2}\right)=-71.5651^\circ=108.4349^\circ$

Be careful $\begin{cases} x<0 \\ y>0 \end{cases}$

$z=z=0$

$\Rightarrow R(\rho,\varphi,z)=(6.3246,\ 108.4349^\circ,\ 0)$

$R(r,\theta,\varphi)=?$

$r=\sqrt{x^2+y^2+z^2}=\sqrt{(-2)^2+6^2+0^2}=\sqrt{40}=6.3246$

$\theta=\tan^{-1}\left(\frac{\rho}{z}\right)=\tan^{-1}\left(\frac{\sqrt{40}}{0}\right)=90^\circ$

$\varphi=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{6}{-2}\right)=108.4349^\circ$

$\Rightarrow R(r,\theta,\varphi)=(6.3246,\ 90^\circ,\ 108.4349^\circ)$

2.7 Convert these vectors to Cylindrical & Spherical systems ?

a) $$\underline{F} = \frac{x\underline{a}_x + y\underline{a}_y + 4\underline{a}_z}{\sqrt{x^2+y^2+z^2}}$$

$$\begin{bmatrix} F_\rho \\ F_\varphi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

$$\begin{cases} F_x = \dfrac{x}{\sqrt{x^2+y^2+z^2}} = \dfrac{\rho\cos\varphi}{\sqrt{\rho^2+z^2}} \\[2ex] F_y = \dfrac{y}{\sqrt{x^2+y^2+z^2}} = \dfrac{\rho\sin\varphi}{\sqrt{\rho^2+z^2}} \\[2ex] F_z = \dfrac{4}{\sqrt{x^2+y^2+z^2}} = \dfrac{4}{\sqrt{\rho^2+z^2}} \end{cases}$$

$$\begin{cases} F_\rho = F_x\cos\varphi + F_y\sin\varphi = \dfrac{\rho\cos^2\varphi + \rho\sin^2\varphi}{\sqrt{\rho^2+z^2}} = \dfrac{\rho}{\sqrt{\rho^2+z^2}} \\[2ex] F_\varphi = -F_x\sin\varphi + F_y\cos\varphi = \dfrac{-\rho\cos\varphi\sin\varphi + \rho\sin\varphi\cos\varphi}{\sqrt{\rho^2+z^2}} = 0 \\[2ex] F_z = \dfrac{4}{\sqrt{\rho^2+z^2}} \end{cases}$$

$$\Rightarrow \underline{F} = \frac{\rho\underline{a}_\rho + 4\underline{a}_z}{\sqrt{\rho^2+z^2}} \quad \text{(Cylindrical)}$$

$$\begin{bmatrix} F_r \\ F_\theta \\ F_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & \theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

2.7

a)

$$\begin{cases} F_x = \dfrac{x}{\sqrt{x^2+y^2+z^2}} = \dfrac{r\sin\theta\cos\varphi}{r} = \sin\theta\cos\varphi \\[2ex] F_y = \dfrac{y}{\sqrt{x^2+y^2+z^2}} = \dfrac{r\sin\theta\sin\varphi}{r} = \sin\theta\sin\varphi \\[2ex] F_z = \dfrac{4}{\sqrt{x^2+y^2+z^2}} = \dfrac{4}{r} \end{cases}$$

$$\begin{cases} F_r = F_x\sin\theta\cos\varphi + F_y\sin\theta\sin\varphi + F_z\cos\theta \\ \quad = \sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \frac{4}{r}\cos\theta = \sin^2\theta + \frac{4}{r}\cos\theta \\ F_\theta = F_x\cos\theta\cos\varphi + F_y\cos\theta\sin\varphi - F_z\sin\theta \\ \quad = \sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi - \frac{4}{r}\sin\theta = \sin\theta\cos\theta - \frac{4}{r}\sin\theta \\ F_\varphi = -F_x\sin\varphi + F_y\cos\varphi \\ \quad = -\sin\theta\cos\varphi\sin\varphi + \sin\theta\sin\varphi\cos\varphi = 0 \end{cases}$$

$$\Rightarrow \underline{F} = \left(\sin^2\theta + \frac{4}{r}\cos\theta\right)\underline{a}_r + \left(\sin\theta\cos\theta - \frac{4}{r}\sin\theta\right)\underline{a}_\theta \quad \text{(Spherical)}$$

b) $$\underline{G} = (x^2+y^2)\left[\frac{x\,\underline{a}_x}{\sqrt{x^2+y^2+z^2}} + \frac{y\,\underline{a}_y}{\sqrt{x^2+y^2+z^2}} + \frac{z\,\underline{a}_z}{\sqrt{x^2+y^2+z^2}}\right]$$

$$\begin{cases} G_x = \dfrac{\rho^2(\rho\cos\varphi)}{\sqrt{\rho^2+z^2}} \\[2ex] G_y = \dfrac{\rho^2(\rho\sin\varphi)}{\sqrt{\rho^2+z^2}} \\[2ex] G_z = \dfrac{\rho^2 z}{\sqrt{\rho^2+z^2}} \end{cases}$$

2.7

b)

$$\begin{cases} G_\rho = G_x \cos\varphi + G_y \sin\varphi = \dfrac{\rho^3\cos^2\varphi + \rho^3\sin^2\varphi}{\sqrt{\rho^2+z^2}} = \dfrac{\rho^3}{\sqrt{\rho^2+z^2}} \\ G_\varphi = -G_x \sin\varphi + G_y \cos\varphi = \dfrac{-\rho^3\cos\varphi\sin\varphi + \rho^3\sin\varphi\cos\varphi}{\sqrt{\rho^2+z^2}} = 0 \\ G_z = \dfrac{\rho^2 z}{\sqrt{\rho^2+z^2}} \end{cases}$$

$$\Rightarrow \underline{G} = \frac{\rho^2(\rho\underline{a}_\rho + z\underline{a}_z)}{\sqrt{\rho^2+z^2}} \quad (\text{Cylindrical})$$

$$\begin{cases} G_x = \dfrac{(r^2\sin^2\theta)(r\sin\theta\cos\varphi)}{r} = r^2\sin^3\theta\cos\varphi \\ G_y = \dfrac{(r^2\sin^2\theta)(r\sin\theta\sin\varphi)}{r} = r^2\sin^3\theta\sin\varphi \\ G_z = \dfrac{(r^2\sin^2\theta)(r\cos\theta)}{r} = r^2\sin^2\theta\cos\theta \end{cases}$$

$$\begin{cases} G_r = G_x\sin\theta\cos\varphi + G_y\sin\theta\sin\varphi + G_z\cos\theta \\ \quad = r^2\sin^4\theta\cos^2\varphi + r^2\sin^4\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\theta \\ \quad = r^2\sin^4\theta + r^2\sin^2\theta\cos^2\theta = r^2\sin^2\theta \\ G_\theta = G_x\cos\theta\cos\varphi + G_y\cos\theta\sin\varphi - G_z\sin\theta \\ \quad = r^2\sin^3\theta\cos\varphi\cos\theta\cos\varphi + r^2\sin^3\theta\sin\varphi\cos\theta\sin\varphi - r^2\sin^3\theta\cos\theta \\ \quad = r^2\sin^3\theta\cos\theta - r^2\sin^3\theta\cos\theta = 0 \\ G_\varphi = -G_x\sin\varphi + G_y\cos\varphi \\ \quad = -r^2\sin^3\theta\cos\varphi\sin\varphi + r^2\sin^3\theta\sin\varphi\cos\varphi = 0 \end{cases}$$

$$\Rightarrow \underline{G} = (r^2\sin^2\theta)\underline{a}_r \quad (\text{Spherical})$$

2.13 Prove ?

a) $\underline{a}_x \cdot \underline{a}_\rho \overset{?}{=} \cos\varphi$ $\qquad$ $\underline{a}_y \cdot \underline{a}_\rho \overset{?}{=} \sin\varphi$

$\underline{a}_x \cdot \underline{a}_\varphi \overset{?}{=} -\sin\varphi$ $\qquad$ $\underline{a}_y \cdot \underline{a}_\varphi \overset{?}{=} \cos\varphi$

In cylindrical system :

$$\begin{cases} \underline{a}_\rho = \cos\varphi\, \underline{a}_x + \sin\varphi\, \underline{a}_y \\ \underline{a}_\varphi = -\sin\varphi\, \underline{a}_x + \cos\varphi\, \underline{a}_y \end{cases}$$

Therefore :

$$\underline{a}_x \cdot \underline{a}_\rho = \underline{a}_x \cdot (\cos\varphi\, \underline{a}_x + \sin\varphi\, \underline{a}_y) = \cos\varphi\, \underbrace{\underline{a}_x \cdot \underline{a}_x}_{=1} + \sin\varphi\, \underbrace{\underline{a}_x \cdot \underline{a}_y}_{=0} = \cos\varphi$$

$$\underline{a}_x \cdot \underline{a}_\varphi = \underline{a}_x \cdot (-\sin\varphi\, \underline{a}_x + \cos\varphi\, \underline{a}_y) = -\sin\varphi$$

$$\underline{a}_y \cdot \underline{a}_\rho = \underline{a}_y \cdot (\cos\varphi\, \underline{a}_x + \sin\varphi\, \underline{a}_y) = \cos\varphi\, \underbrace{\underline{a}_y \cdot \underline{a}_x}_{=0} + \sin\varphi\, \underbrace{\underline{a}_y \cdot \underline{a}_y}_{=1} = \sin\varphi$$

$$\underline{a}_y \cdot \underline{a}_\varphi = \underline{a}_y \cdot (-\sin\varphi\, \underline{a}_x + \cos\varphi\, \underline{a}_y) = \cos\varphi$$

b, c) $\underline{a}_x \cdot \underline{a}_r \overset{?}{=} \sin\theta\cos\varphi$ $\qquad$ $\underline{a}_y \cdot \underline{a}_r \overset{?}{=} \sin\theta\sin\varphi$ $\qquad$ $\underline{a}_z \cdot \underline{a}_r \overset{?}{=} \cos\theta$

$\underline{a}_x \cdot \underline{a}_\theta \overset{?}{=} \cos\theta\cos\varphi$ $\qquad$ $\underline{a}_y \cdot \underline{a}_\theta \overset{?}{=} \cos\theta\sin\varphi$ $\qquad$ $\underline{a}_z \cdot \underline{a}_\theta \overset{?}{=} -\sin\theta$

In spherical system :

$$\begin{cases} \underline{a}_r = \sin\theta\cos\varphi\, \underline{a}_x + \sin\theta\sin\varphi\, \underline{a}_y + \cos\theta\, \underline{a}_z \\ \underline{a}_\theta = \cos\theta\cos\varphi\, \underline{a}_x + \cos\theta\sin\varphi\, \underline{a}_y - \sin\theta\, \underline{a}_z \end{cases}$$

So :

$$\underline{a}_x \cdot \underline{a}_r = \sin\theta\cos\varphi\, \underbrace{\underline{a}_x \cdot \underline{a}_x}_{=1} + \sin\theta\sin\varphi\, \underbrace{\underline{a}_x \cdot \underline{a}_y}_{=0} + \cos\theta\, \underbrace{\underline{a}_x \cdot \underline{a}_z}_{=0} = \sin\theta\cos\varphi$$

$$\underline{a}_x \cdot \underline{a}_\theta = \cos\theta\cos\varphi\, \underbrace{\underline{a}_x \cdot \underline{a}_x}_{=1} + \cos\theta\sin\varphi\, \underbrace{\underline{a}_x \cdot \underline{a}_y}_{=0} - \sin\theta\, \underbrace{\underline{a}_x \cdot \underline{a}_z}_{=0} = \cos\theta\cos\varphi$$

Similarly :

$\underline{a}_y \cdot \underline{a}_r = \sin\theta\sin\varphi$ $\qquad$ $\underline{a}_z \cdot \underline{a}_r = \cos\theta$

$\underline{a}_y \cdot \underline{a}_\theta = \cos\theta\sin\varphi$ $\qquad$ $\underline{a}_z \cdot \underline{a}_\theta = -\sin\theta$

2.15 $\underline{F} = 10\,\underline{a}_x - 4\,\underline{a}_y + 2\,\underline{a}_z$

Express $\underline{F}$ using these base vectors:

$$\begin{cases} \underline{a}_1 = \frac{1}{\sqrt{2}}(\underline{a}_x - \underline{a}_z) \\ \underline{a}_2 = \frac{1}{\sqrt{2}}(\underline{a}_x + \underline{a}_z) \\ \underline{a}_3 = \frac{1}{1}\,\underline{a}_y \end{cases}$$

$$\underline{F} = F_1\,\underline{a}_1 + F_2\,\underline{a}_2 + F_3\,\underline{a}_3$$

$$\begin{cases} F_1 = \underline{a}_1 \cdot \underline{F} = \frac{1}{\sqrt{2}}(\underline{a}_x - \underline{a}_z) \cdot (10\,\underline{a}_x - 4\,\underline{a}_y + 2\,\underline{a}_z) \\ \quad = \frac{10}{\sqrt{2}} - 0 - \frac{2}{\sqrt{2}} = \frac{10-2}{\sqrt{2}} = \frac{8}{\sqrt{2}} \\ F_2 = \underline{a}_2 \cdot \underline{F} = \frac{1}{\sqrt{2}}(\underline{a}_x + \underline{a}_z) \cdot (10\,\underline{a}_x - 4\,\underline{a}_y + 2\,\underline{a}_z) \\ \quad = \frac{10}{\sqrt{2}} - 0 + \frac{2}{\sqrt{2}} = \frac{10+2}{\sqrt{2}} = \frac{12}{\sqrt{2}} \\ F_3 = \underline{a}_3 \cdot \underline{F} = \underline{a}_y \cdot (10\,\underline{a}_x - 4\,\underline{a}_y + 2\,\underline{a}_z) \\ \quad = -4 \end{cases}$$

$$\Rightarrow \underline{F} = \frac{8}{\sqrt{2}}\,\underline{a}_1 + \frac{12}{\sqrt{2}}\,\underline{a}_2 - 4\,\underline{a}_3$$

2.17 $\underline{A} = 3\underline{a}_\rho + 2\underline{a}_\varphi + \underline{a}_z$

$\underline{B} = 5\underline{a}_\rho + 0\,\underline{a}_\varphi - 8\underline{a}_z$

a) $\underline{A} + \underline{B} = ?$

$= (3+5)\underline{a}_\rho + (2+0)\underline{a}_\varphi + (1-8)\underline{a}_z$

$= 8\underline{a}_\rho + 2\underline{a}_\varphi - 7\underline{a}_z$

b) $\underline{A} \cdot \underline{B} = ?$

$= 3\times 5 + 2\times 0 + 1\times(-8)$

$= 7$

c) $\underline{A} \times \underline{B} = ?$

$$= \begin{vmatrix} \underline{a}_\rho & \underline{a}_\varphi & \underline{a}_z \\ 3 & 2 & 1 \\ 5 & 0 & -8 \end{vmatrix} = (-16-0)\underline{a}_\rho - (-24-5)\underline{a}_\varphi + (0-10)\underline{a}_z$$

$= -16\underline{a}_\rho + 29\,\underline{a}_\varphi - 10\,\underline{a}_z$

d) The angle between $\underline{A}$ & $\underline{B}$ = ?

$\underline{A} \cdot \underline{B} = |\underline{A}||\underline{B}| \cos\theta_{AB}$

$\Rightarrow \theta_{AB} = \cos^{-1}\left(\dfrac{\underline{A}\cdot\underline{B}}{|\underline{A}||\underline{B}|}\right)$

$\underline{A}\cdot\underline{B} = 7$

$|\underline{A}| = \sqrt{3^2+2^2+1^2} = \sqrt{14}$

$|\underline{B}| = \sqrt{5^2+0+(-8)^2} = \sqrt{89}$

$\Rightarrow \theta_{AB} = \cos^{-1}\left(\dfrac{7}{\sqrt{14}\sqrt{89}}\right) = 78.562^\circ$

2.18 $\quad \underline{A} = \rho \cos\varphi \, \underline{a}_\rho + \rho z^2 \sin\varphi \, \underline{a}_z$

a) $\underline{A}$ in rectangular coordinates = ?

$|\underline{A}|$ @ $(3,-4,0)$ = ?

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix} \qquad \begin{cases} A_\rho = \rho\cos\varphi \\ A_\varphi = 0 \\ A_z = \rho z^2 \sin\varphi \end{cases}$$

$$\begin{cases} A_x = \rho\cos^2\varphi \\ A_y = \rho\cos\varphi\sin\varphi \\ A_z = \rho z^2 \sin\varphi \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \rho\cos\varphi = x \ , \ \rho\sin\varphi = y \\ \cos\varphi = \dfrac{x}{\rho} = \dfrac{x}{\sqrt{x^2+y^2}} \\ \sin\varphi = \dfrac{y}{\rho} = \dfrac{y}{\sqrt{x^2+y^2}} \end{cases}$$

$$\Rightarrow \begin{cases} A_x = \dfrac{x^2}{\sqrt{x^2+y^2}} \\ A_y = \dfrac{xy}{\sqrt{x^2+y^2}} \\ A_z = y z^2 \end{cases}$$

$$\Rightarrow \underline{A} = \frac{x^2}{\sqrt{x^2+y^2}} \underline{a}_x + \frac{xy}{\sqrt{x^2+y^2}} \underline{a}_y + y z^2 \underline{a}_z$$

$$\underline{A} \text{ at } (3,-4,0) = \frac{3^2}{\sqrt{9+16}} \underline{a}_x + \frac{3\times(-4)}{\sqrt{9+16}} \underline{a}_y + (-4)\times 0 \, \underline{a}_z$$

$$= \frac{9}{5} \underline{a}_x - \frac{12}{5} \underline{a}_y$$

$$|\underline{A}| \text{ at } (3,-4,0) = \sqrt{\left(\frac{9}{5}\right)^2 + \left(-\frac{12}{5}\right)^2} = \sqrt{\frac{225}{25}} = 3$$

2.18

b) $\underline{A}$ in spherical system = ?

$|\underline{A}|$ @ $(3,-4,0)=?$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{cases} A_x = \dfrac{x^2}{\sqrt{x^2+y^2}} = \dfrac{x^2}{\rho} = \dfrac{(r\sin\theta\cos\varphi)^2}{r\sin\theta} = r\sin\theta\cos^2\varphi \\ A_y = \dfrac{xy}{\sqrt{x^2+y^2}} = \dfrac{xy}{\rho} = \dfrac{(r\sin\theta\cos\varphi)(r\sin\theta\sin\varphi)}{r\sin\theta} = r\sin\theta\cos\varphi\sin\varphi \\ A_z = yz^2 = (r\sin\theta\sin\varphi)(r^2\cos^2\theta) = r^3\sin\theta\cos^2\theta\sin\varphi \end{cases}$$

$\Rightarrow$

$$\begin{cases} A_r = A_x\sin\theta\cos\varphi + A_y\sin\theta\sin\varphi + A_z\cos\theta \\ \quad = r\sin^2\theta\cos^3\varphi + r\sin^2\theta\cos\varphi\sin^2\varphi + r^3\sin\theta\cos^3\theta\sin\varphi \\ \quad = r\sin^2\theta\cos\varphi + r^3\sin\theta\cos^3\theta\sin\varphi \\ A_\theta = A_x\cos\theta\cos\varphi + A_y\cos\theta\sin\varphi - A_z\sin\theta \\ \quad = r\sin\theta\cos\theta\cos^3\varphi + r\sin\theta\cos\theta\cos\varphi\sin^2\varphi - r^3\sin^2\theta\cos^2\theta\sin\varphi \\ \quad = r\sin\theta\cos\theta\cos\varphi - r^3\sin^2\theta\cos^2\theta\sin\varphi \\ A_\varphi = -A_x\sin\varphi + A_y\cos\varphi \\ \quad = -r\sin\theta\cos^2\varphi\sin\varphi + r\sin\theta\cos^2\varphi\sin\varphi \\ \quad = 0 \end{cases}$$

$$\Rightarrow \underline{A} = (r\sin^2\theta\cos\varphi + r^3\sin\theta\cos^3\theta\sin\varphi)\,\underline{a}_r$$
$$+ (r\sin\theta\cos\theta\cos\varphi - r^3\sin^2\theta\cos^2\theta\sin\varphi)\,\underline{a}_\theta$$

2.18

b)

$\underline{A}$ at $(3,-4,0) = ?$

$$\hookrightarrow \begin{cases} r = \sqrt{3^2+(-4)^2} = 5 \\ \theta = \tan^{-1}\left(\frac{\rho}{z}\right) = \tan^{-1}\left(\frac{5}{0}\right) = 90^\circ \Rightarrow \begin{cases} \sin\theta = 1 \\ \cos\theta = 0 \end{cases} \\ \varphi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-4}{3}\right) = 306.8699^\circ \Rightarrow \begin{cases} \cos\varphi = \frac{x}{\rho} = \frac{3}{5} \\ \sin\varphi = \frac{y}{\rho} = \frac{-4}{5} \end{cases} \end{cases}$$

$$\underline{A} = \left(5\times 1^2\times \frac{3}{5} + 5^3\times 1\times 0^3\times\left(\frac{-4}{5}\right)\right)\underline{a}_r + \left(5\times 1\times 0\times\frac{3}{5} - 5^3\times 1^2\times 0^2\times\left(\frac{-4}{5}\right)\right)\underline{a}_\theta$$

$$= 3\,\underline{a}_r$$

$|\underline{A}|$ at $(3,-4,0) = 3$

2.21 Distance between these pair of points ?

a) (2, 1, 5) and (6, −1, 2)

$$\underline{r} = (6-2)\underline{a}_x + (-1-1)\underline{a}_y + (2-5)\underline{a}_z$$

$$= 4\underline{a}_x - 2\underline{a}_y - 3\underline{a}_z$$

$$d = |\underline{r}| = \sqrt{4^2 + (-2)^2 + (-3)^2} = \sqrt{29} = 5.3852$$

b) $(3, \frac{\pi}{2}, -1)$ and $(5, \frac{3\pi}{2}, 5)$

$$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\varphi_2 - \varphi_1) + (z_2 - z_1)^2}$$

$$= \sqrt{5^2 + 3^2 - 2 \times 3 \times 5 \times \underbrace{\cos\left(\frac{3\pi}{2} - \frac{\pi}{2}\right)}_{=-1} + (5-(-1))^2}$$

$$= \sqrt{25 + 9 + 30 + 36} = \sqrt{100}$$

$$= 10$$

c) $(10, \frac{\pi}{4}, \frac{3\pi}{4})$ and $(5, \frac{\pi}{6}, \frac{7\pi}{4})$

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta_1\cos\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2\cos(\varphi_2 - \varphi_1)}$$

$$= \sqrt{10^2 + 5^2 - 2 \times 10 \times 5 \times \cos\frac{\pi}{4} \times \cos\frac{\pi}{6} - 2 \times 10 \times 5 \times \sin\frac{\pi}{4} \times \sin\frac{\pi}{6} \times \cos\left(\frac{7\pi}{4} - \frac{3\pi}{4}\right)}$$

$$= \sqrt{125 - 100\left(\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} \times (-1)\right)} = \sqrt{125 - 25(\sqrt{6} - \sqrt{2})}$$

$$= 9.9558$$

2.25 $\underline{A} = (2z - \sin\varphi)\underline{a}_\rho + (4\rho + 2\cos\varphi)\underline{a}_\varphi - 3\rho z \underline{a}_z$

$\underline{B} = \rho\cos\varphi\, \underline{a}_\rho + \sin\varphi\, \underline{a}_\varphi + \underline{a}_z$

a) Minimum angle between $\underline{A}$ & $\underline{B}$ at $(1, 60^\circ, -1) = ?$

At $(1, 60^\circ, -1) \Rightarrow \begin{cases} \underline{A} = (-2 - \sin\frac{\pi}{3})\underline{a}_\rho + (4 + 2\cos\frac{\pi}{3})\underline{a}_\varphi - 3\times(-1)\underline{a}_z \\ \quad = -2.8660\, \underline{a}_\rho + 5\underline{a}_\varphi + 3\underline{a}_z \\ \underline{B} = 1\times\cos\frac{\pi}{3}\, \underline{a}_\rho + \sin\frac{\pi}{3}\, \underline{a}_\varphi + \underline{a}_z \\ \quad = \frac{1}{2}\underline{a}_\rho + 0.8660\, \underline{a}_\varphi + \underline{a}_z \end{cases}$

$$\theta_{AB} = \cos^{-1}\left(\frac{\underline{A}.\underline{B}}{|\underline{A}||\underline{B}|}\right)$$

$\underline{A}.\underline{B} = -1.4330 + 4.330 + 3 = 5.8971$

$|\underline{A}| = \sqrt{8.2141 + 25 + 9} = 6.4972$

$|\underline{B}| = \sqrt{0.25 + 0.75 + 1} = 1.4142$

$$\Rightarrow \theta_{AB} = \cos^{-1}\left(\frac{5.8971}{6.4972\times 1.4142}\right) = 50.0744^\circ$$

b) unit vector normal to both $\underline{A}$ & $\underline{B}$ at $(1, 90^\circ, 0) = ?$

At $(1, 90^\circ, 0) \Rightarrow \begin{cases} \underline{A} = (0 - 1)\underline{a}_\rho + (4 + 0)\underline{a}_\varphi - 3\times1\times0\, \underline{a}_z = -\underline{a}_\rho + 4\underline{a}_\varphi \\ \underline{B} = 1\times 0\, \underline{a}_\rho + 1\times\underline{a}_\varphi + \underline{a}_z = \underline{a}_\varphi + \underline{a}_z \end{cases}$

$$\underline{A}\times\underline{B} = \begin{vmatrix} \underline{a}_\rho & \underline{a}_\varphi & \underline{a}_z \\ -1 & 4 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 4\underline{a}_\rho + \underline{a}_\varphi - \underline{a}_z$$

$$\underline{a}^\perp = \frac{\underline{A}\times\underline{B}}{|\underline{A}\times\underline{B}|} = \frac{1}{\sqrt{16+1+1}}(4\underline{a}_\rho + \underline{a}_\varphi - \underline{a}_z) = 0.9428\,\underline{a}_\rho + 0.2357\,\underline{a}_\varphi - 0.2357\,\underline{a}_z$$

2.30 $\underline{H} = 5\rho \sin\varphi \, \underline{a}_\rho - \rho z \cos\varphi \, \underline{a}_\varphi + 2\rho \, \underline{a}_z$

@ Point $(2, 30°, -1)$

a) unit vector along $\underline{H}$ = ?

At $(2, 30°, -1)$ : $\underline{H} = 5 \times 2 \times \sin 30° \, \underline{a}_\rho - 2 \times (-1) \cos 30° \, \underline{a}_\varphi + 2 \times 2 \, \underline{a}_z$

$$= 5 \underline{a}_\rho + \sqrt{3} \, \underline{a}_\varphi + 4 \underline{a}_z$$

$$|\underline{H}| = \sqrt{25 + 3 + 16} = \sqrt{44}$$

$$\Rightarrow \underline{a}_H = \frac{\underline{H}}{|\underline{H}|} = 0.7538 \, \underline{a}_\rho + 0.2611 \, \underline{a}_\varphi + 0.6030 \, \underline{a}_z$$

b) component of $\underline{H}$ parallel to $\underline{a}_x$ = ?

$$H_x = H_\rho \cos\varphi - H_\varphi \sin\varphi$$

$$= 5\rho \sin\varphi \cos\varphi + \rho z \sin\varphi \cos\varphi$$

$H_x$ at $(2, 30°, -1) = 5 \times 2 \times \sin 30 \cos 30 + 2 \times (-1) \sin 30 \cos 30$

$$= 8 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$\underline{H}_x = 2\sqrt{3} \, \underline{a}_x$

c) component of $\underline{H}$ normal to $\rho = 2$ = ?

$H_\rho \underline{a}_\rho$ is the component of $\underline{H}$ normal to $\rho = 2$.

$$\Rightarrow \underline{H}_n = H_\rho \underline{a}_\rho = 5\rho \sin\varphi \, \underline{a}_\rho$$

at $(2, 30°, -1)$ : $\underline{H}_n = 5 \times 2 \times \sin 30° \, \underline{a}_\rho = 5 \underline{a}_\rho$

d) component of $\underline{H}$ tangential to $\varphi = 30°$ = ?

$H_\rho \underline{a}_\rho + H_z \underline{a}_z$ is the component of $\underline{H}$ tangential to $\varphi = 30°$.

$$\Rightarrow \underline{H}_t = H_\rho \underline{a}_\rho + H_z \underline{a}_z = 5\rho \sin\varphi \, \underline{a}_\rho + 2\rho \, \underline{a}_z$$

at $(2, 30°, -1)$ : $\underline{H}_t = 5 \underline{a}_\rho + 4 \underline{a}_z$