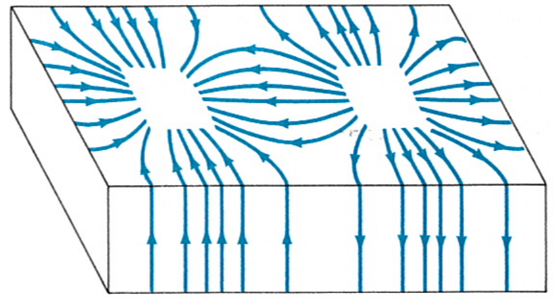

FIGURE 3.27 For Review Question 3.8.

3.8 The surface current density **J** in a rectangular waveguide is plotted in Figure 3.27. It is evident from the figure that **J** diverges at the top wall of the guide, whereas it is divergenceless at the side wall.

(a) True (b) False

3.9 Stokes's theorem is applicable only when a closed path exists and the vector field and its derivatives are continuous within the path.

(a) True (c) Not necessarily
(b) False

3.10 If a vector field **Q** is solenoidal, which of these is true?

(a) $\oint_L \mathbf{Q} \cdot d\mathbf{l} = 0$ (d) $\nabla \times \mathbf{Q} \neq 0$
(b) $\oint_S \mathbf{Q} \cdot d\mathbf{S} = 0$ (e) $\nabla^2 \mathbf{Q} = 0$
(c) $\nabla \times \mathbf{Q} = 0$

Answers: 3.1a-(vi), b-(vii), c-(v), d-(i), e-(ii), f-(iv), g-(iii), 3.2a-(vi), b-(v), c-(vii), d-(ii), e-(i), f-(iv), g-(iii), 3.3a-(v), b-(vi), c-(iv), d-(iii), e-(i), f-(ii), 3.4b, 3.5c, 3.6c, 3.7e, 3.8a, 3.9a, 3.10b.

PROBLEMS

Section 3.2—Differential Length, Area, and Volume

3.1 Using the differential length dl, find the length of each of the following curves:

(a) $\rho = 3, \pi/4 < \phi < \pi/2, z = \text{constant}$
(b) $r = 1, \theta = 30°, 0 < \phi < 60°$
(c) $r = 4, 30° < \theta < 90°, \phi = \text{constant}$

3.2 Calculate the areas of the following surfaces using the differential surface area dS:

(a) $\rho = 2, 0 < z < 5, \pi/3 < \phi < \pi/2$
(b) $z = 1, 1 < \rho < 3, 0 < \phi < \pi/4$
(c) $r = 10, \pi/4 < \theta < 2\pi/3, 0 < \phi < 2\pi$
(d) $0 < r < 4, 60° < \theta < 90°, \phi = \text{constant}$

3.3 Use the differential volume dv to determine the volumes of the following regions:

(a) $0 < x < 1, 1 < y < 2, -3 < z < 3$
(b) $2 < \rho < 5, \pi/3 < \phi < \pi, -1 < z < 4$
(c) $1 < r < 3, \pi/2 < \theta < 2\pi/3, \pi/6 < \phi < \pi/2$

3.4 A surface is defined by $\rho = 2, \pi/6 < \phi < \pi/3, 0 < z < 1$. Find the area of the surface.

3.5 Find the volume of a region bounded by $\rho = 1, \rho = 4, \phi = 0, \phi = \pi, z = 0$, and $z = 2$.

Sections 3.3—Line, Surface, and Volume Integrals

3.6 Given that $\mathbf{H} = x^2\mathbf{a}_x + y^2\mathbf{a}_y$, evaluate $\int_L \mathbf{H} \cdot d\mathbf{l}$, where L is along the curve $y = x^2$ from $(0, 0)$ to $(1, 1)$.

3.7 Given that $\rho_s = x^2 + xy$, calculate $\int_S \rho_s dS$ over the region $y \leq x^2, 0 < x < 1$.

3.8 If the integral $\int_A^B \mathbf{F} \cdot d\mathbf{l}$ is regarded as the work done in moving a particle from A to B, find the work done by the force field

$$\mathbf{F} = 2xy\mathbf{a}_x + (x^2 - z^2)\mathbf{a}_y - 3xz^2\mathbf{a}_z$$

on a particle that travels from $A(0, 0, 0)$ to $B(2, 1, 3)$ along

(a) The segment $(0, 0, 0) \rightarrow (0, 1, 0) \rightarrow (2, 1, 0) \rightarrow (2, 1, 3)$
(b) The straight line $(0, 0, 0)$ to $(2, 1, 3)$

3.9 A vector field is represented by $\mathbf{F} = \rho^2\mathbf{a}_\rho + z\mathbf{a}_\phi + \cos\phi\mathbf{a}_z$ Newtons. Evaluate the work done or $\int_L \mathbf{F} \cdot d\mathbf{l}$, where L is from $P(2, 0°, 0)$ to $Q(2, \pi/4, 3)$. Assume that L consists of the arc $\rho = 2, 0 < \phi < \pi/4, z = 0$, followed by the line $\rho = 2, \phi = \pi/4, 0 < z < 3$.

3.10 (a) Express $\mathbf{F} = \dfrac{-y}{x^2 + y^2}\mathbf{a}_x + \dfrac{x}{x^2 + y^2}\mathbf{a}_y$ in cylindrical coordinate system.

(b) Evaluate $\oint_L \mathbf{F} \cdot d\mathbf{l}$, where L is the circular path $x^2 + y^2 = 4$.

3.11 If

$$\mathbf{H} = (x - y)\mathbf{a}_x + (x^2 + zy)\mathbf{a}_y + 5yz\mathbf{a}_z$$

evaluate $\int \mathbf{H} \cdot d\mathbf{l}$ along the contour of Figure 3.28.

Sections 3.5—Gradient of a Scalar

3.12 Calculate the gradient of:

(a) $V_1 = 6xy - 2xz + z$
(b) $V_2 = 10\rho\cos\phi - \rho z$
(c) $V_3 = \dfrac{2}{r}\cos\phi$

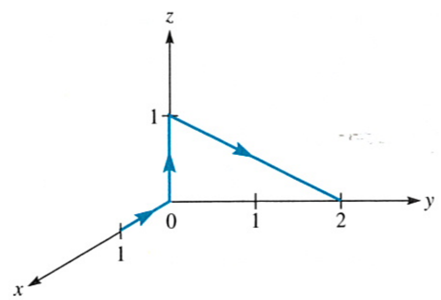

FIGURE 3.28 For Problem 3.7.

3.13 Determine the gradients of the following fields:

(a) $U = e^{x+2y} \cosh z$

(b) $T = \dfrac{3z}{\rho} \cos \phi$

(c) $W = \dfrac{5 \cos \phi}{r} + 2r^2 \sin \phi$

3.14 If $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is the position vector of point (x, y, z), $r = |\mathbf{r}|$, and n is an integer, show that $\nabla r^n = nr^{n-2}\mathbf{r}$.

3.15 The temperature in an auditorium is given by $T = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ in the auditorium desires to fly in such a direction that it will get warm as soon as possible. In what direction must it fly?

3.16 A family of planes is described by $F = x - 2y + z$. Find a unit normal $\mathbf{a}_n$ to the planes.

Sections 3.6—Divergence of a Vector and Divergence Theorem

3.17 Evaluate the divergence of the following vectors fields:

(a) $\mathbf{A} = xy\mathbf{a}_x + y^2\mathbf{a}_y - xz\mathbf{a}_z$

(b) $\mathbf{B} = \rho z^2\mathbf{a}_\rho + \rho \sin^2 \phi \, \mathbf{a}_\phi + 2\rho z \sin^2 \phi \, \mathbf{a}_z$

(c) $\mathbf{C} = r\mathbf{a}_r + r \cos^2 \theta \, \mathbf{a}_\phi$

3.18 The heat flow vector $\mathbf{H} = k\nabla T$, where T is the temperature and k is the thermal conductivity. Show that where

$$T = 50 \sin \frac{\pi x}{2} \cosh \frac{\pi y}{2}$$

then $\nabla \cdot \mathbf{H} = 0$.

3.19 (a) Prove that

$$\nabla \cdot (V\mathbf{A}) = V\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla V$$

where V is a scalar field and $\mathbf{A}$ is a vector field.

(b) Evaluate $\nabla \cdot (V\mathbf{A})$ when $\mathbf{A} = 2x\mathbf{a}_x + 3y\mathbf{a}_y - 4z\mathbf{a}_z$ and $V = xyz$.

3.20 If $U = xz - x^2y + y^2z^2$, evaluate div (grad U).

3.21 If $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ and $\mathbf{T} = 2zy\mathbf{a}_x + xy^2\mathbf{a}_y + x^2yz\mathbf{a}_z$, determine

(a) $(\nabla \cdot \mathbf{r})\mathbf{T}$

(b) $(\mathbf{r} \cdot \nabla)\mathbf{T}$

(c) $\nabla \cdot \mathbf{r}(\mathbf{r} \cdot \mathbf{T})$

(d) $(\mathbf{r} \cdot \nabla)r^2$

3.22 If $\mathbf{A} = 2x\mathbf{a}_x - z^2\mathbf{a}_y + 3xy\mathbf{a}_z$, find the flux of $\mathbf{A}$ through a surface defined by $\rho = 2$, $0 < \phi < \pi/2$, $0 < z < 1$.

3.23 Let $\mathbf{D} = 2\rho z^2\mathbf{a}_\rho + \rho \cos^2 \phi \, \mathbf{a}_z$. Evaluate

(a) $\oint_S \mathbf{D} \cdot d\mathbf{S}$

(b) $\int_v \nabla \cdot \mathbf{D} dv$

over the region defined by $2 \leq \rho \leq 5$, $-1 \leq z \leq 1$, $0 < \phi < 2\pi$.

3.24 If $\mathbf{H} = 10 \cos \theta \mathbf{a}_r$, evaluate $\int \mathbf{H} \cdot d\mathbf{S}$ over a hemisphere defined by $r = 1$, $0 < \phi < 2\pi$, $0 < \theta < \pi/2$.

3.25 (a) Given that $\mathbf{A} = xy\mathbf{a}_x + yz\mathbf{a}_y + xz\mathbf{a}_z$, evaluate $\oint_S \mathbf{A} \cdot d\mathbf{S}$, where S is the surface of the cube defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

(b) Repeat part (a) if S remains the same but $\mathbf{A} = yz\mathbf{a}_x + xz\mathbf{a}_y + xy\mathbf{a}_z$.

3.26 Evaluate both sides of the divergence theorem for the vector field

$$\mathbf{H} = 2xy\mathbf{a}_x + (x^2 + z^2)\mathbf{a}_y + 2yz\mathbf{a}_z$$

and the rectangular region defined by $0 < x < 1$, $1 < y < 2$, $-1 < z < 3$.

3.27 Given that $\mathbf{B} = \rho\mathbf{a}_\rho + 10z\mathbf{a}_z$ evaluate both sides of the divergence theorem for the region defined by $0 \leq \rho \leq 3$, $0 \leq \phi \leq 2\pi$, $0 \leq z \leq 4$.

3.28 Let $\mathbf{F} = 10\rho \sin \phi\mathbf{a}_\rho + 4\rho \cos \phi\mathbf{a}_\phi + z\mathbf{a}_z$. Evaluate both sides of the divergence theorem for the region defined by $\rho = 0$, $\rho = 1$, $\phi = 0$, $\phi = \pi/2$, $z = -1$, and $z = 2$.

3.29 Verify the divergence theorem for the function $\mathbf{A} = r^2\mathbf{a}_r + r \sin \theta \cos \phi \, \mathbf{a}_\theta$ over the surface of a quarter of a hemisphere defined by $0 < r < 3$, $0 < \phi < \pi/2$, $0 < \theta < \pi/2$.

3.30 Calculate the total outward flux of vector

$$\mathbf{F} = \rho^2 \sin \phi \, \mathbf{a}_\rho + z \cos \phi \, \mathbf{a}_\phi + \rho z\mathbf{a}_z$$

through the hollow cylinder defined by $2 \leq \rho \leq 3$, $0 \leq z \leq 5$.

Sections 3.7—Curl of a Vector and Stokes's Theorem

3.31 Evaluate the curl of the following vector fields:

(a) $\mathbf{A} = xy\mathbf{a}_x + y^2\mathbf{a}_y - xz\mathbf{a}_z$

(b) $\mathbf{B} = \rho z^2\mathbf{a}_\rho + \rho \sin^2\phi\, \mathbf{a}_\phi + 2\rho z \sin^2\phi\, \mathbf{a}_z$

(c) $\mathbf{C} = r\mathbf{a}_r + r\cos^2\theta\, \mathbf{a}_\phi$

3.32 Evaluate $\nabla \times \mathbf{A}$ and $\nabla \cdot (\nabla \times \mathbf{A})$ if:

(a) $\mathbf{A} = x^2y\mathbf{a}_x + y^2z\mathbf{a}_y - 2xz\mathbf{a}_z$

(b) $\mathbf{A} = \rho^2z\mathbf{a}_\rho + \rho^3\mathbf{a}_\phi + 3\rho z^2\mathbf{a}_z$

(c) $\mathbf{A} = \dfrac{\sin\phi}{r^2}\mathbf{a}_r - \dfrac{\cos\phi}{r^2}\mathbf{a}_\theta$

***3.33** Given that $\mathbf{F} = x^2y\mathbf{a}_x - y\mathbf{a}_y$, find

(a) $\oint_L \mathbf{F}\cdot d\mathbf{l}$ where L is shown in Figure 3.29.

(b) $\int_S (\nabla \times \mathbf{F})\cdot d\mathbf{S}$, where S is the area bounded by L.

(c) Is Stokes's theorem satisfied?

3.34 Let $\mathbf{A} = \rho \sin\phi\, \mathbf{a}_\rho + \rho^2\mathbf{a}_\phi$, evaluate $\oint_L \mathbf{A}\cdot d\mathbf{l}$ if L is the contour of Figure 3.30.

3.35 If $\mathbf{F} = 2xy\mathbf{a}_x + y\mathbf{a}_y$, evaluate $\oint_L \mathbf{F}\cdot d\mathbf{l}$ around L shown in Figure 3.31.

3.36 If $\mathbf{F} = 2\rho z\mathbf{a}_\rho + 3z\sin\phi\, \mathbf{a}_\phi - 4\rho\cos\phi\, \mathbf{a}_z$, verify Stokes's theorem for the open surface defined by $z = 1, 0 < \rho < 2, 0 < \phi < 45°$.

3.37 Let $\mathbf{A} = 4x^2e^{-y}\mathbf{a}_x - 8xe^{-y}\mathbf{a}_y$. Determine $\nabla \times [\nabla(\nabla \cdot A)]$.

3.38 Let $V = \dfrac{\sin\theta\cos\phi}{r}$. Determine:

(a) ∇V, (b) $\nabla \times \nabla V$, (c) $\nabla \cdot \nabla V$

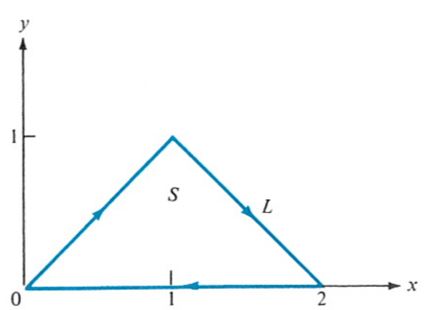

FIGURE 3.29 For Problem 3.31.

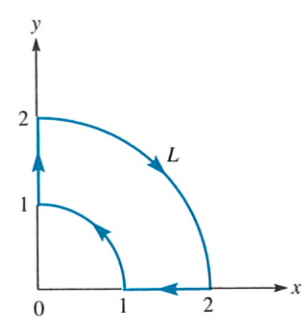

FIGURE 3.30 For Problem 3.32.

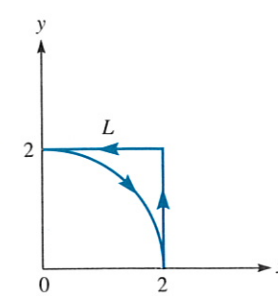

FIGURE 3.31 For Problem 3.33.

****3.39** A vector field is given by

$$\mathbf{Q} = \frac{\sqrt{x^2 + y^2 + z^2}}{\sqrt{x^2 + y^2}}[(x - y)\mathbf{a}_x + (x + y)\mathbf{a}_y]$$

Evaluate the following integrals:

(a) $\int_L \mathbf{Q}\cdot d\mathbf{l}$, where L is the circular edge of the volume in the form of an ice cream cone shown in Figure 3.32.

(b) $\int_{S_1} (\nabla \times \mathbf{Q})\cdot d\mathbf{S}$, where S_1 is the top surface of the volume

(c) $\int_{S_2} (\nabla \times \mathbf{Q})\cdot d\mathbf{S}$, where S_2 is the slanting surface of the volume

(d) $\int_{S_1} \mathbf{Q}\cdot d\mathbf{S}$

(e) $\int_{S_2} \mathbf{Q}\cdot d\mathbf{S}$

(f) $\int_v \nabla\cdot\mathbf{Q}\, dv$

How do your results in parts (a) to (f) compare?

***3.40** A rigid body spins about a fixed axis through its center with angular velocity ω. If $\mathbf{u}$ is the velocity at any point in the body, show that $\omega = 1/2\, \nabla \times \mathbf{u}$.

3.41 Find the divergence and curl of the following vectors:

(a) $\mathbf{H} = 3\rho z\mathbf{a}_\rho - 4\rho\cos\phi\mathbf{a}_z$

(b) $\mathbf{G} = r\mathbf{a}_r + r\sin\theta\mathbf{a}_\theta$

3.42 For a vector field $\mathbf{A}$ and a scalar field V, show in Cartesian coordinates that

(a) $\nabla\cdot(V\nabla V) = V\nabla^2 V + |\nabla V|^2$

(b) $\nabla \times (V\mathbf{A}) = V\nabla \times \mathbf{A} + \nabla V \times \mathbf{A}$

3.43 Given that $\mathbf{F} = (2z^2 - 5y)\mathbf{a}_x + 6xz^2\mathbf{a}_y + (2x + y)\mathbf{a}_z$, find $\nabla \times \nabla \times \mathbf{F}$ at (1, 2, –3).

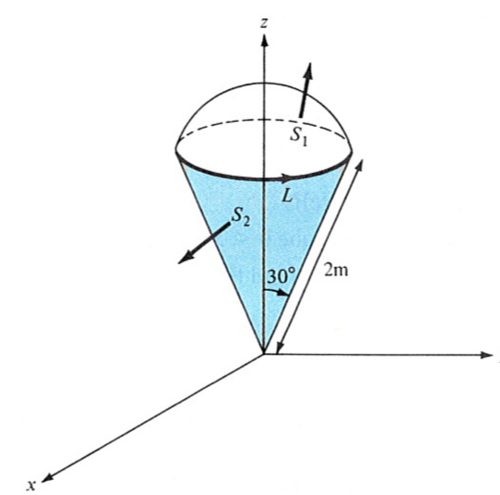

FIGURE 3.32 Volume in form of ice cream cone for Problem 3.38.

**Double asterisks indicate problems of highest difficulty.

Sections 3.8—Laplacian of a Scalar

3.44 Find $\nabla^2 V$ for each of the following scalar fields:

(a) $V_1 = x^3 + y^3 + z^3$
(b) $V_2 = \rho z^2 \sin 2\phi$
(c) $V_3 = r^2(1 + \cos\theta \sin\phi)$

3.45 Find the Laplacian of the following scalar fields and compute the value at the specified point.

(a) $U = x^3 y^2 e^{xz}$, $(1, -1, 1)$
(b) $V = \rho^2 z(\cos\phi + \sin\phi)$, $(5, \pi/6, -2)$
(c) $W = e^{-r}\sin\theta\cos\phi$, $(1, \pi/3, \pi/6)$

3.46 If $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is the position vector of point (x, y, z), $r = |\mathbf{r}|$, and n is an integer, show that:

(a) $\nabla(\ln r) = \dfrac{\mathbf{r}}{r^2}$

(b) $\nabla^2(\ln r) = \dfrac{1}{r^2}$

3.47 Let $V = xy^2z^3$, evaluate ∇V and $\nabla^2 V$ at point $P(1, 2, 3)$.

3.48 If $V = \dfrac{5\cos\phi}{r^2}$, find: (a) ∇V, (b) $\nabla \cdot \nabla V$, (c) $\nabla \times \nabla V$.

***3.49** In cylindrical coordinates,

$$\nabla^2\mathbf{A} = \left(\nabla^2 A_\rho - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi} - \frac{A_\rho}{\rho^2}\right)\mathbf{a}_\rho + \left(\nabla^2 A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi} - \frac{A_\phi}{\rho^2}\right)\mathbf{a}_\phi + \nabla^2 A_z\mathbf{a}_z$$

If $\mathbf{G} = 2\rho\sin\phi\mathbf{a}_\rho + 4\rho\cos\phi\mathbf{a}_\phi + (z^2 + 1)\rho\mathbf{a}_z$, find $\nabla^2\mathbf{G}$.

Sections 3.9—Classification of Vector Fields

3.50 Given the vector field

$$\mathbf{G} = (16xy - z)\mathbf{a}_x + 8x^2\mathbf{a}_y - x\mathbf{a}_z$$

(a) Is $\mathbf{G}$ irrotational (or conservative)?
(b) Find the net flux of $\mathbf{G}$ over the cube $0 < x, y, z < 1$.
(c) Determine the circulation of $\mathbf{G}$ around the edge of the square $z = 0$, $0 < x, y < 1$. Assume anticlockwise direction.

3.51 If the vector field

$$\mathbf{T} = (\alpha xy + \beta z^3)\mathbf{a}_x + (3x^2 - \gamma z)\mathbf{a}_y + (3xz^2 - y)\mathbf{a}_z$$

is irrotational, determine α, β, and γ. Find $\nabla \cdot \mathbf{T}$ at $(2, -1, 0)$.

3.52 Show that the vector field $\mathbf{F} = yz\mathbf{a}_x + xz\mathbf{a}_y + xy\mathbf{a}_z$ is both solenoidal and conservative.

PART

2

ELECTROSTATICS