Prob 2.30

(a)

At P, $\rho = 2,\ \phi = 30°,\ z = -1$

$$\mathbf{H} = 10\sin 30\,\mathbf{a}_\rho + 2\cos 30°\,\mathbf{a}_\phi + 4\mathbf{a}_z$$

$$= 5\mathbf{a}_\rho + 1.732\,\mathbf{a}_\phi + 4\,\mathbf{a}_z.$$

$$\mathbf{a}_H = \frac{(5,\ 1.732,\ 4)}{\sqrt{5^2 + 1.732^2 + 4^2}} = \underline{0.7538\,\mathbf{a}_\rho + 0.2611\,\mathbf{a}_\phi + 0.603\,\mathbf{a}_z}.$$

(b) $H_x = H_\rho\cos\phi - H_\phi\sin\phi = 5\rho\sin\phi\cos\phi + \rho z\cos\phi\sin\phi$

or P at $\rho = 2,\ \phi = 30,\ z = -1;$

$$\mathbf{H}_x = H_x\mathbf{a}_x = (10\sin 30°\cos 30° - 2\sin 30°\cos 30°)\mathbf{a}_x = 8\sin 30°\cos 30°\,\mathbf{a}_x$$

$$= \underline{3.4641\mathbf{a}_x}$$

(c) Normal to $\rho = 2$ is $\mathbf{H}_n = H_\rho\,\mathbf{a}_\rho = 10\sin\phi\,\mathbf{a}_\rho$;

i.e. $\mathbf{H}_n = \underline{5\,\mathbf{a}_\rho}$.

(d) Tangential to $\phi = 30°$.

$$\mathbf{H}_t = H_\rho\,\mathbf{a}_\rho + H_z\mathbf{a}_z = \underline{5\mathbf{a}_\rho + 4\mathbf{a}_z}$$

Prob. 2.31

(a) $5 = \mathbf{r}\cdot\mathbf{a}_x + \mathbf{r}\cdot\mathbf{a}_y = x + y$ $\quad$ a plane

(b) $$10 = |\mathbf{r}\times\mathbf{a}_z| = \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \end{vmatrix} = |y\mathbf{a}_x - x\mathbf{a}_y| = \sqrt{x^2 + y^2} = \rho$$

a cylinder of infinite length

CHAPTER 3

P. E. 3.1

(a) $$DH = \int_{\phi=45°}^{\phi=60°} r\sin\theta\, d\phi \Big|_{r=3,90°} = 3(1)\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{4} = \underline{0.7854.}$$

(b) $$FG = \int_{\theta=60°}^{\theta=90°} r d\theta \Big|_{r=5} = 5\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{5\pi}{6} = \underline{2.618.}$$

(c)

$$AEHD = \int_{\theta=60°}^{\theta=90°}\int_{\phi=45°}^{\phi=60°} r^2\sin\theta\, d\theta\, d\phi \Big|_{r=3} = 9(-\cos\theta)\Big|_{\theta=60°}^{\theta=90°}\ \phi\Big|_{\phi=45°}^{\phi=60°}$$

$$= 9\left(\frac{1}{2}\right)\left(\frac{\pi}{12}\right) = \frac{3\pi}{8} = \underline{1.178.}$$

(d)

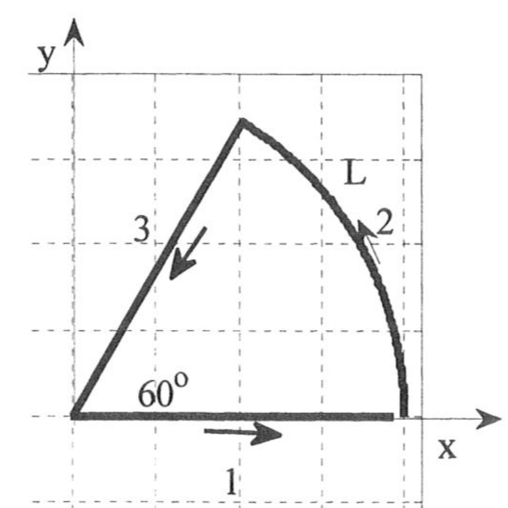
$$ABCD = \int_{r=3}^{r=5}\int_{\theta=60}^{\theta=90} r d\theta\, dr = \frac{r^2}{2}\Big|_{r=3}^{r=5}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{4\pi}{3} = \underline{4.189.}$$

(e)

$$\text{Volume} = \int_{r=3}^{r=5}\int_{\phi=45°}^{\phi=60°}\int_{\theta=60}^{\theta=90} r^2\sin\theta dr\, d\theta\, d\phi = \frac{r^3}{3}\Big|_{r=3}^{r=5}(-\cos\theta)\Big|_{\theta=60°}^{\theta=90°}\ \phi\Big|_{\phi=45°}^{\phi=60°} = \frac{1}{3}(98)\left(\frac{1}{2}\right)\frac{\pi}{12}$$

$$= \underline{\frac{49\pi}{36} = 4.276}.$$

P.E. 3.2



$$\oint_L A\bullet dl = \left(\int_1 + \int_2 + \int_3\right) A\bullet dl = C_1 + C_2 + C_3$$

Along (1), $C_1 = \int A\bullet dl = \int_0^2 \rho\cos\phi\, d\rho\,|_{\phi=0} = \frac{\rho^2}{2}\Big|_0^2 = 2.$

Along (2), $dl = \rho d\phi\, a_\phi$, $A\bullet dl = 0$, $C_2 = 0$

Along (3), $C_3 = \int_2^0 \rho\cos\phi\, d\rho_{\phi=60°} = -\frac{\rho^2}{2}\Big|_0^2 \left(\frac{1}{2}\right) = -1$

$$\oint_l A\bullet dl = C_1 + C_2 + C_3 = 2 + 0 - 1 = \underline{\underline{1}}$$

P.E. 3.3

(a) $\nabla U = \frac{\partial U}{\partial x} a_x + \frac{\partial U}{\partial y} a_y + \frac{\partial U}{\partial z} a_z$

$$= \underline{\underline{y(2x+z)a_x + x(x+z)a_y + xy\, a_z}}$$

(b) $\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z$

$$= \underline{\underline{(z\sin\phi + 2\rho)a_\rho + (z\cos\phi - \frac{z^2}{\rho}\sin 2\phi)a_\phi + (\rho\sin\phi + 2z\cos^2\phi)a_z}}$$

(c)

$$\nabla f = \frac{\partial f}{\partial r} a_r + \frac{1}{r}\frac{\partial f}{\partial \theta} a_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} a_\phi$$

$$= \left(\frac{\cos\theta\sin\phi}{r} + 2r\phi\right)a_r - \frac{\sin\theta\sin\phi\ln r}{r} a_\theta + \frac{(\cos\theta\cos\phi\ln r + r^2)}{r\sin\theta} a_\phi$$

$$= \underline{\underline{\left(\frac{\cos\theta\sin\phi}{r} + 2r\phi\right)a_r - \frac{\sin\theta\sin\phi\ln r}{r} a_\theta + \left(\frac{\cot\theta\cos\phi\ln r}{r} + r\,\mathrm{cosec}\theta\right)a_\phi}}$$

P.E. 3.4

$\nabla\Phi = (y+z)a_x + (x+z)a_y + (x+y)a_z$

At (1,2,3), $\nabla\Phi = \underline{\underline{(5,4,3)}}$

$\nabla\Phi \bullet a_l = (5,4,3)\bullet\frac{(2,2,1)}{3} = \frac{21}{3} = \underline{\underline{7}},$

where $(2,2,1) = (3,4,4) - (1,2,3)$

P.E. 3.5

Let $f = x^2y + z - 3$, $g = x\log z - y^2 + 4$,

$\nabla f = 2xy\, a_x + x^2 a_y + a_z,$

$\nabla g = \log z\, a_x - 2y\, a_y + \frac{x}{z} a_z$

At P(−1,2,1),

$n_f = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{(-4a_x + a_y + a_z)}{\sqrt{18}}$, $n_g = \pm\frac{\nabla g}{|\nabla g|} = \pm\frac{(-4a_y - a_z)}{\sqrt{17}}$

$\cos\theta = n_f . n_g = \pm\frac{(-5)}{\sqrt{18\times 17}}$

Take positive value to get acute angle.

$\theta = \cos^{-1}\frac{5}{17.493} = \underline{\underline{73.39°}}$

P.E. 3.6

(a) $\nabla\bullet A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 + 4x + 0 = \underline{\underline{4x}}.$

At (1,−2,3), $\nabla\bullet A = \underline{\underline{4}}.$

(b)

$$\nabla\bullet B = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial\rho}$$

$$= \frac{1}{\rho}2\rho z\sin\phi - \frac{1}{\rho}3\rho z^2\sin\phi = 2z\sin\phi - 3z^2\sin\phi$$

$$= \underline{\underline{(2-3z)z\sin\phi}}.$$

At $(5,\frac{\pi}{2},1)$, $\nabla\bullet B = (2-3)(1) = \underline{\underline{-1}}.$

(c)

$$\nabla\bullet C = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 C_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(C_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial C_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}6r^2\cos\theta\cos\phi$$

$$= \underline{\underline{6\cos\theta\cos\phi}}$$

$At\,(1,\frac{\pi}{6},\frac{\pi}{3}),\quad \nabla \bullet \boldsymbol{C} = 6\cos\frac{\pi}{6}\cos\frac{\pi}{3} = \underline{\underline{2.598.}}$

P.E. 3.7 This is similar to Example 3.7.

$$\Psi = \oint_S \boldsymbol{D} \bullet d\boldsymbol{S} = \Psi_t + \Psi_b + \Psi_c$$

$\Psi_t = 0 = \Psi_b$ since $\boldsymbol{D}$ has no z-component

$$\Psi_c = \iint \rho^2 \cos^2\phi \rho d\phi dz = \rho^3 \int_{\phi=0}^{\phi=2\pi} \cos^2\phi d\phi \int_{z=0}^{z=1} dz \Big|_{\rho=4}$$

$$= (4)^3 \pi (1) = 64\pi$$

$$\Psi = 0 + 0 + 64\pi = \underline{\underline{64\pi}}$$

By the divergence theorem,

$$\oint_S \boldsymbol{D} \bullet d\boldsymbol{S} = \oint_V \nabla \bullet \boldsymbol{D} dv$$

$$\nabla \bullet \boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^3\cos^2\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi} z\sin\phi + \frac{\partial A_z}{dz}$$

$$= 3\rho\cos^2\phi + \frac{z}{\rho}\cos\phi.$$

$$\Psi = \int_V \nabla \bullet \boldsymbol{D} dv = \int_V (3\rho\cos^2\phi + \frac{z}{\rho}\cos\phi)\rho d\phi dz d\rho$$

$$= 3\int_0^4 \rho^2 d\rho \int_0^{2\pi} \cos^2\phi d\phi \int_0^1 dz + \int_0^4 d\rho \int_0^{2\pi} \cos\phi d\phi \int_0^1 z dz$$

$$= 3(\frac{4^3}{3})\pi(1) = \underline{\underline{64\pi.}}$$

P.E. 3.8

(a)

$$\nabla \times \boldsymbol{A} = \boldsymbol{a}_x(1-0) + \boldsymbol{a}_y(y-0) + \boldsymbol{a}_z(4y-z)$$

$$= \underline{\underline{\boldsymbol{a}_x + y\boldsymbol{a}_y + (4y-z)\boldsymbol{a}_z}}$$

At $(1,-2,3), \nabla \times \boldsymbol{A} = \underline{\underline{\boldsymbol{a}_x - 2\boldsymbol{a}_y - 11\boldsymbol{a}_z}}$

(b)

$$\nabla \times \boldsymbol{B} = \boldsymbol{a}_\rho(0 - 6\rho z\cos\phi) + \boldsymbol{a}_\phi(\rho\sin\phi - 0) + \boldsymbol{a}_z\frac{1}{\rho}(6\rho z^2\cos\phi - \rho z\cos\phi)$$

$$= \underline{\underline{-6\rho z\cos\phi\,\boldsymbol{a}_\rho + \rho\sin\phi\,\boldsymbol{a}_\phi + (6z-1)z\cos\phi\,\boldsymbol{a}_z}}$$

At $(5,\frac{\pi}{2},-1), \nabla \times \boldsymbol{B} = \underline{\underline{5\boldsymbol{a}_\phi}}$

(c)

$$\nabla \times \boldsymbol{C} = \boldsymbol{a}_r\frac{1}{r\sin\theta}(r^{-1/2}\cos\theta - 0) + \frac{\boldsymbol{a}_\theta}{r}(-\frac{2r\cos\theta\sin\phi}{\sin\theta} - \frac{3}{2}r^{1/2}) + \frac{\boldsymbol{a}_\phi}{r}(0 + 2r\sin\theta\cos\phi)$$

$$= \underline{\underline{r^{-1/2}\cot\theta\,\boldsymbol{a}_r - (2\cot\theta\sin\phi + \frac{3}{2}r^{-1/2})\boldsymbol{a}_\theta + 2\sin\theta\cos\phi\,\boldsymbol{a}_\phi}}$$

At $(1,\frac{\pi}{6},\frac{\pi}{3}),\quad \nabla \times \boldsymbol{C} = \underline{\underline{1.732\boldsymbol{a}_r - 4.5\boldsymbol{a}_\theta + 0.5\boldsymbol{a}_\phi}}$

P.E. 3.9

$$\oint_L \boldsymbol{A} \bullet d\boldsymbol{l} = \int_S (\nabla \times \boldsymbol{A}) \bullet d\boldsymbol{S}$$

But $(\nabla \times \boldsymbol{A}) = \sin\phi\,\boldsymbol{a}_z + \frac{z\cos\phi}{\rho}\boldsymbol{a}_\rho$ and $dS = \rho d\phi d\rho\,\boldsymbol{a}_z$

$$\int_S (\nabla \times \boldsymbol{A}) \bullet d\boldsymbol{S} = \iint \rho\sin\phi\, d\phi\, d\rho$$

$$= \frac{\rho^2}{2}\Big|_0^2 (-\cos\phi)\Big|_0^{60°}$$

$$= 2(-\frac{1}{2}+1) = \underline{\underline{1.}}$$

P.E. 3.10

$$\nabla\times\nabla V = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} =$$

$$= (\frac{\partial^2 V}{\partial y\partial z} - \frac{\partial^2 V}{\partial y\partial z})a_x + (\frac{\partial^2 V}{\partial x\partial z} - \frac{\partial^2 V}{\partial z\partial x})a_y + (\frac{\partial^2 V}{\partial x\partial y} - \frac{\partial^2 V}{\partial y\partial x})a_z = 0$$

P.E. 3.11

(a)

$$\nabla^2 U = \frac{\partial}{\partial x}(2xy + yz) + \frac{\partial}{\partial y}(x^2 + xz) + \frac{\partial}{\partial z}(xy)$$

$$= \underline{\underline{2y.}}$$

(b)

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho(z\sin\phi + 2\rho) + \frac{1}{\rho^2}(-\rho z\sin\phi - 2z^2\frac{\partial}{\partial\rho}\sin\phi\cos\phi) + \frac{\partial}{\partial z}(\rho\sin\phi + 2z\cos^2\phi)$$

$$= \frac{1}{\rho}(z\sin\phi + 4\rho) - \frac{1}{\rho^2}(z\rho\sin\phi + 2z^2\cos 2\phi) + 2\cos^2\phi.$$

$$= \underline{\underline{4 + 2\cos^2\phi - \frac{2z^2}{\rho^2}\cos 2\phi.}}$$

(c)

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\frac{1}{r}\cos\theta\sin\phi + 2r^3\phi] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}[-\sin^2\theta\sin\phi\ln r]$$

$$+ \frac{1}{r^2\sin^2\theta}[-\cos\theta\sin\phi\ln r]$$

$$= \underline{\underline{\frac{1}{r^2}\cos\theta\sin\phi(1 - 2\ln r - \csc^2\theta\ln r) + 6\phi}}$$

P.E. 3.12

If $\boldsymbol{B}$ is conservative, $\nabla\times\boldsymbol{B} = \boldsymbol{0}$ must be satisfied.

$$\nabla\times\boldsymbol{B} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z\cos xz & x & x\cos xz \end{vmatrix}$$

$$= 0a_x + (\cos xz - xz\sin xz - \cos xz + xz\sin xz)a_y + (1-1)a_z = \boldsymbol{0}$$

Hence <u>B is a conservative field</u>.

Prob. 3.1

(a)

$$dl = \rho d\phi; \qquad \rho = 3$$

$$L = \int dl = 3\int_{\pi/4}^{\pi/2} d\phi = 3(\frac{\pi}{2} - \frac{\pi}{4}) = \frac{3\pi}{4} = \underline{\underline{2.356}}$$

(b)

$$dl = r\sin\theta d\phi; \qquad r = 1, \quad \theta = 30°;$$

$$L = \int dl = r\sin\theta\int_0^{\pi/3} d\phi = (1)\sin 30°[(\frac{\pi}{3}) - 0] = \underline{\underline{0.5236.}}$$

(c)

$$dl = rd\theta$$

$$L = \int dl = r\int_{\pi/6}^{\pi/2} d\theta = 4(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{4\pi}{3} = \underline{\underline{4.189}}$$

Prob. 3.2

(a)

$$dS = \rho d\phi dz$$

$$S = \int dS = \rho\iint d\phi dz = 2\int_0^5 dz \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\phi = 2(5)[\frac{\pi}{2} - \frac{\pi}{3}] = \frac{10\pi}{6} = \underline{\underline{5.236}}$$

(b)

In cylindrical, $dS = \rho d\rho d\phi$

$$S = \int dS = \int_1^3 \rho d\rho \int_0^{\frac{\pi}{4}} d\phi = \frac{\pi}{4}(\frac{\rho^2}{2})\Big|_1^3 = \underline{\underline{3.142}}$$

(c) In spherical, $dS = r^2 \sin\theta\, d\phi\, d\theta$

$$S = \int dS = 100 \int_{\frac{\pi}{4}}^{\frac{2\pi}{3}} \sin\theta d\theta \int_0^{2\pi} d\phi = 100(2\pi)(-\cos\theta)\Big|_{\frac{\pi}{4}}^{\frac{2\pi}{3}} = 200\pi(0.5+0.7071) = \underline{\underline{758.4}}$$

(d)

$$dS = r\,dr\,d\theta$$

$$S = \int dS = \int_0^4 r dr \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta = \frac{r^2}{2}\Big|_0^4 (\frac{\pi}{2} - \frac{\pi}{3}) = \frac{8\pi}{6} = \underline{\underline{4.189}}$$

Prob.3.3

(a) $dV = dxdydz$

$$V = \int dxdydz = \int_0^1 dx \int_1^2 dy \int_{-3}^3 dz = (1)\ (2-1)(3--3) = \underline{\underline{6}}$$

(b) $dV = \rho d\phi d\rho dz$

$$V = \int_2^5 \rho d\rho \int_{-1}^4 dz \int_{\frac{\pi}{3}}^{\pi} d\phi = \frac{\rho^2}{2}\Big|_2^5 (4--1)(\pi - \frac{\pi}{3}) = \frac{1}{2}(25-4)(5)(\frac{2\pi}{3}) = 35\pi = \underline{\underline{110}}$$

(c) $dV = r^2 \sin\theta\, dr d\theta\, d\phi$

$$V = \int_1^3 r^2 dr \int_{\frac{\pi}{2}}^{2\frac{\pi}{3}} \sin\theta \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\phi = \frac{r^3}{3}\Big|_1^3 (-\cos\theta)\Big|_{\pi/2}^{\pi/3}(\frac{\pi}{2} - \frac{\pi}{6})$$

$$= \frac{1}{3}(27-1)(\frac{1}{2})(\frac{\pi}{3}) = \frac{26\pi}{18} = \underline{\underline{4.538}}$$

Prob.3.4

$$S = \int dS = \int_{z=0}^{1}\int_{\phi=\pi/6}^{\pi/3} \rho d\phi dz\Big|_{\rho=2}$$

$$= 2\int_0^1 dz \int_{\pi/6}^{\pi/3} d\phi = 2(1)(\pi/3 - \pi/6) = \frac{\pi}{3} = \underline{\underline{1.0472}}$$

Prob.3.5

$$V = \int_v dv = \int_{z=0}^{2}\int_{\phi=0}^{\pi}\int_{\rho=1}^{4} \rho d\rho d\phi dz$$

$$= \int_0^2 dz \int_0^{\pi} d\phi \int_1^4 \rho d\rho = (2)(\pi)(\frac{\rho^2}{2}\Big|_1^4) = 2\pi\frac{1}{2}(16-1) = 15\pi = \underline{\underline{47.12}}$$

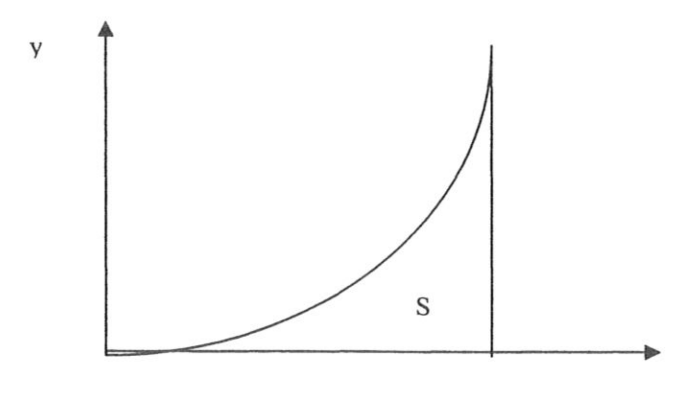
Prob. 3.6

$$\int_L \boldsymbol{H} \bullet d\boldsymbol{l} = \int(x^2 dx + y^2\, dy)$$

But on L, $y = x^2 \quad dy = 2xdx$

$$\int_L \boldsymbol{H} \bullet d\boldsymbol{l} = \int_0^1 (x^2 + x^4.2x)dx = \frac{x^3}{3} + 2\frac{x^6}{6}\Big|_0^1 = \frac{1}{3} + \frac{1}{3} = \underline{\underline{0.6667}}$$

Prob 3.7



$$\int \rho_s dS = \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}(x^2+xy)dy\,dx$$

$$= \int_0^1 (x^2 y + \frac{xy^2}{2}\Bigg|_{y=0}^{x^2})dx = \int_0^1 (x^4+\frac{x^5}{2})dx$$

$$= \frac{1}{5}+\frac{1}{12} = \frac{17}{60} = \underline{\underline{0.2833}}$$

Prob. 3.8

(a)

$$\int F \bullet dl = \int_{y=0}^{1}(x^2-z^2)dy\Big|_{x=0,z=0} + \int_{x=0}^{x=2} 2xydx\Big|_{y=1,z=0} + \int_{z=0}^{z=3}(-3xz^2)dz\Big|_{x=2,y=1}$$

$$= 0+2(1)\frac{x^2}{2}\Big|_0^2 - 3(2)\frac{z^3}{3}\Big|_0^3$$

$$= 0+4-54 = \underline{\underline{-50}}$$

(b)

Let $x=2t.\ \ y=t,\ \ z=3t$

$dx = 2dt,\ \ dy = dt,\ \ dz = 3dt;$

$$\int F \bullet dl = \int_0^1 (8t^2-5t^2-162t^3)\ dt$$

$$= (t^3-40.5t^4)\Big|_0^1 = \underline{\underline{-39.5}}$$

Prob.3.9

$$W = \int_L F \bullet dl = \int_{\phi=0}^{\pi/4} z\rho d\phi\ \Big|_{z=0,\rho=2} + \int_{z=0}^{3}\rho\cos\phi dz\Big|_{\rho=2,\phi=\pi/4}$$

$$= 0+2\cos(\pi/4)(3) = 6\cos 45^o\ \underline{\underline{= 4.243\text{ J}}}$$

Prob. 3.10

(a)
$$\begin{bmatrix}F_\rho\\F_\phi\\F_z\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}F_x\\F_y\\F_z\end{bmatrix}$$

$y=\rho\sin\phi, x=\rho\cos\phi$

$$F_\rho = F_x\cos\phi + F_y\sin\phi = \frac{-y}{x^2+y^2}\cos\phi + \frac{x}{x^2+y^2}\sin\phi$$

$$= -\frac{\rho\sin\phi\cos\phi}{\rho^2}+\frac{\rho\cos\phi\sin\phi}{\rho^2} = 0$$

$$F_\phi = -F_x\sin\phi + F_y\cos\phi = \frac{y}{x^2+y^2}\sin\phi + \frac{x}{x^2+y^2}\cos\phi$$

$$= \frac{\rho\sin^2\phi}{\rho^2}+\frac{\rho\cos^2\phi}{\rho^2} = \frac{1}{\rho}$$

$F_z = 0$

$$\underline{\underline{\boldsymbol{F} = \frac{1}{\rho}\boldsymbol{a}_\phi}}$$

$dl = \rho d\phi \boldsymbol{a}_\phi,\qquad \rho = 2$

(b)
$$\oint F \bullet dl = \int_0^{2\pi}\frac{1}{\rho}\rho d\phi = \int_0^{2\pi} d\phi = 2\pi = \underline{\underline{6.283}}$$

Prob. 3.11

$$\int H \bullet dl = \int_{x=1}^{0}(x-y)dx\Big|_{y=0,z=0} + \int_{z=0}^{1}5yzdz\Big|_{x=0,y=0}$$

$$+\int(x^2+zy)dy + 5yzdz\Big|_{x=0,z=1-y/2}$$

$$= \int_1^0 xdx + \int_0^2 (y-\frac{y^2}{2})dy + \int_1^0 (10z-10z^2)dz$$

$$= \underline{\underline{-1.5}}$$

Prob. 3.12

(a)
$$\nabla V_1 = \frac{\partial V_1}{\partial x}\boldsymbol{a}_x + \frac{\partial V_1}{\partial y}\boldsymbol{a}_y + \frac{\partial V_1}{\partial z}\boldsymbol{a}_z$$

$$= \underline{\underline{(6y-2z)\boldsymbol{a}_x + 6x\boldsymbol{a}_y + (1-2x)\boldsymbol{a}_z}}$$

(b) $\nabla V_2 = \frac{\partial V_2}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V_2}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V_2}{\partial z}\boldsymbol{a}_z$

$$= \underline{\underline{(10\cos\phi - z)\boldsymbol{a}_\rho - 10\sin\phi\boldsymbol{a}_\phi - \rho\boldsymbol{a}_z}}$$

(c)

$$\nabla V_3 = \frac{\partial V_3}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial V_3}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V_3}{\partial \phi}\boldsymbol{a}_\phi$$

$$= -\frac{2}{r^2}\cos\phi\boldsymbol{a}_r + 0 + \frac{1}{r\sin\theta}\left(-\frac{2}{r}\sin\phi\right)\boldsymbol{a}_\phi$$

$$= \underline{\underline{-\frac{2}{r^2}\cos\phi\boldsymbol{a}_r - \frac{2\sin\phi}{r^2\sin\theta}\boldsymbol{a}_\phi}}$$

Prob. 3.13

(a)

$$\nabla U = \frac{\partial U}{\partial x}\boldsymbol{a}_x + \frac{\partial U}{\partial y}\boldsymbol{a}_y + \frac{\partial U}{\partial z}\boldsymbol{a}_z$$

$$= \underline{\underline{e^{x+2y}\cosh z\boldsymbol{a}_x + 2e^{x+2y}\cosh z\boldsymbol{a}_y + e^{x+2y}\sinh z\boldsymbol{a}_z}}$$

(b)

$$\nabla T = \frac{\partial T}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial T}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial T}{\partial z}\boldsymbol{a}_z$$

$$= \underline{\underline{-\frac{3z}{\rho^2}\cos\phi\boldsymbol{a}_\rho - \frac{3z}{\rho^2}\sin\phi\boldsymbol{a}_\phi + \frac{3}{\rho}\cos\phi\boldsymbol{a}_z}}$$

(c)

$$\nabla W = \frac{\partial W}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\boldsymbol{a}_\phi$$

$$= \underline{\underline{\left(-\frac{5\cos\theta}{r^2} + 4r\sin\phi\right)\boldsymbol{a}_r + \left[2r^2\cos\phi - \frac{5\sin\phi}{r}\right]\frac{1}{r\sin\theta}\boldsymbol{a}_\phi}}$$

Prob. 3.14

$r = \sqrt{x^2+y^2+z^2}, \qquad r^n = (x^2+y^2+z^2)^{n/2}$

Method 1:

$$\nabla r^n = \frac{\partial r^n}{\partial x}\boldsymbol{a}_x + \frac{\partial r^n}{\partial y}\boldsymbol{a}_y + \frac{\partial r^n}{\partial z}\boldsymbol{a}_z = \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x)\boldsymbol{a}_x + \cdots$$

$$= n(x^2+y^2+z^2)^{(n-2)/2}(x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z) = \underline{\underline{nr^{n-2}\boldsymbol{r}}}$$

Method 2:

$$\nabla r^n = \frac{\partial r^n}{\partial r}\boldsymbol{a}_r = nr^{n-1}\frac{\boldsymbol{r}}{r} = nr^{n-2}\boldsymbol{r}$$

Prob. 3.15

$\nabla T = 2x\boldsymbol{a}_x + 2y\boldsymbol{a}_y - \mathrm{a}_z$

At $(1,1,2), \nabla T = (2,2,-1)$. The mosquito should move in the direction of

$\underline{\underline{2\boldsymbol{a}_x + 2\boldsymbol{a}_y - \boldsymbol{a}_z}}$

Prob. 3.16

$\nabla F = \boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z$

$$a_n = \frac{\nabla F}{|\nabla F|} = \frac{\boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z}{\sqrt{1+4+1}} = \underline{\underline{0.4082\boldsymbol{a}_x - 0.8165\boldsymbol{a}_y + 0.4082\boldsymbol{a}_z}}$$

Prob. 3.17

(a) $\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \underline{\underline{3y - x}}$

(b)

$$\nabla \cdot \boldsymbol{B} = \frac{1}{\rho}2\rho z^2 + \frac{1}{\rho}\rho 2\sin\phi\cos\phi + 2\rho\sin^2\phi$$

$$= \underline{\underline{2z^2 + \sin 2\phi + 2\rho\sin^2\phi}}$$

(c) $\nabla \cdot \mathbf{C} = \frac{1}{r^2}3r^2 + 0 = \underline{\underline{3}}$

Prob. 3.18

$$\nabla\bullet \boldsymbol{H} = k\nabla\bullet\nabla T = k\nabla^2 T$$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 50\sin\frac{\pi x}{2}\cosh\frac{\pi y}{2}\left(-\frac{\pi^2}{4} + \frac{\pi^2}{4}\right) = 0$$

Hence, $\nabla\bullet \boldsymbol{H} = 0$

Prob. 3.19

(a)

$$\nabla\bullet(V A) = \frac{\partial}{\partial x}(V A_x) + \frac{\partial}{\partial y}(V A_y) + \frac{\partial}{\partial z}(V A_z)$$

$$= \left(A_x\frac{\partial V}{\partial x} + V\frac{\partial A_x}{\partial x}\right) + \left(A_y\frac{\partial V}{\partial y} + V\frac{\partial A_y}{\partial y}\right) + \left(A_z\frac{\partial V}{\partial z} + V\frac{\partial A_z}{\partial z}\right)$$

$$= V\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + A_x\frac{\partial V}{\partial x} + A_y\frac{\partial V}{\partial y} + A_z\frac{\partial V}{\partial z}$$

$$= \underline{\underline{V\nabla\bullet A + A\bullet\nabla V}}$$

(b)

$$\nabla\bullet A = 2 + 3 - 4 = 1; \quad \nabla V = yz\,\boldsymbol{a}_x + xz\,\boldsymbol{a}_y + xy\,\boldsymbol{a}_z$$

$$\nabla\bullet(V A) = V\nabla\bullet A + A\bullet\nabla V$$

$$= xyz + 2xyz + 3xyz - 4xyz = \underline{\underline{2xyz}}$$

Prob. 3.20

$$grad\ U = \frac{\partial U}{\partial x}\boldsymbol{a}_x + \frac{\partial U}{\partial y}\boldsymbol{a}_y + \frac{\partial U}{\partial z}\boldsymbol{a}_z$$

$$= (z - 2xy)\boldsymbol{a}_x + (2yz^2 - x^2)\boldsymbol{a}_y + (x + 2y^2 z)\boldsymbol{a}_z$$

$$Div\ grad U = \nabla\bullet\nabla U = \frac{\partial}{\partial x}(z - 2xy) + \frac{\partial}{\partial y}(2yz^2 - x^2) + \frac{\partial}{\partial z}(x + 2y^2 z)$$

$$= -2y + 2z^2 + 2y^2$$

$$= \underline{\underline{2(z^2 + y^2 - y)}}$$

Prob. 3.21

(a)

$$(\nabla\bullet \boldsymbol{r})\boldsymbol{T} = 3\boldsymbol{T} = \underline{\underline{6yz\,\boldsymbol{a}_y + 3xy^2\,\boldsymbol{a}_y + 3x^2 yz\,\boldsymbol{a}_z}}$$

(b)

$$x\frac{\partial \boldsymbol{T}}{\partial x} + y\frac{\partial \boldsymbol{T}}{\partial y} + z\frac{\partial \boldsymbol{T}}{\partial z} = x\,(y^2\,\boldsymbol{a}_y + 2xyz\,\boldsymbol{a}_z) + y(2z\,\boldsymbol{a}_x + 2xy\,\boldsymbol{a}_y + x^2 z\,\boldsymbol{a}_z)$$

$$+ z(2y\,\boldsymbol{a}_x + x^2 y\,\boldsymbol{a}_z)$$

$$= \underline{\underline{4yz\,\boldsymbol{a}_x + 3xy^2\,\boldsymbol{a}_y + 4x^2 yz\,\boldsymbol{a}_z}}$$

(c)

$$\nabla\bullet \boldsymbol{r}(\boldsymbol{r}\bullet\boldsymbol{T}) = 3\,(2xyz + xy^3 + x^2yz^2)$$

$$= \underline{\underline{6xyz + 3xy^3 + 3x^2yz^2}}$$

(d)

$$(\boldsymbol{r}\bullet\nabla)r^2 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$$

$$= x(2x) + y(2y) + z(2z)$$

$$= \underline{\underline{2(x^2 + y^2 + z^2) = 2r^2}}$$

Prob. 3.22

We convert A to cylindrical coordinates; only the ρ-component is needed.

$$A_\rho = A_x\cos\phi + A_y\sin\phi = 2x\cos\phi - z^2\sin\phi$$

But $x = \rho\cos\phi$,

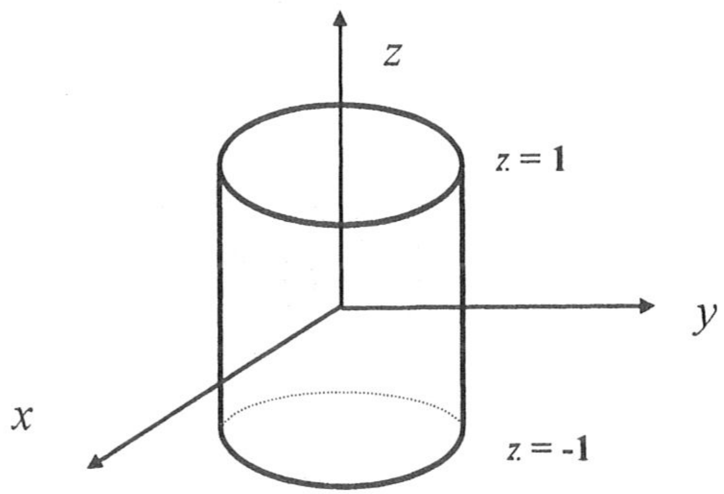
$$A_\rho = 2\rho\cos^2\phi - z^2\sin\phi$$

$$\Psi = \int_S A\cdot dS = \iint A_\rho \rho\, d\phi\, dz = \iint\left[2\rho^2\cos^2\phi - \rho z^2\sin\phi\right]d\phi\, dz$$

$$= 2(2)^2\int_0^{\pi/2}\frac{1}{2}(1 + \cos 2\phi)\,d\phi\int_0^1 dz - 2\int_0^1 z^2 dz\int_0^{\pi/2}\sin\phi\, d\phi$$

$$= 4\left(\phi + \frac{1}{2}\sin 2\phi\right)\Big|_0^{\pi/2} - 2\frac{z^3}{3}\Big|_0^1(-\cos\phi)\Big|_0^{\pi/2} = 2\pi - 2/3 = \underline{\underline{5.6165}}$$

Prob. 3.23



(a)

$$\oint \mathbf{D} \bullet d\mathbf{S} = [\iint_{z=-1} + \iint_{z=1} + \iint_{\rho=5}] \mathbf{D} \bullet d\mathbf{S}$$

$$= -\iint \rho^2 \cos^2\phi d\phi d\rho + \iint \rho^2 \cos^2\phi d\phi d\rho + \iint 2\rho^2 z^2 d\phi dz \Big|_{\rho=5}$$

$$= 2(5)^2 \int_0^{2\pi} d\phi \int_{-1}^{1} z^2 dz = +50(2\pi)(\frac{z^3}{3}\Big|_{-1}^{1})$$

$$= \frac{200\pi}{3} = \underline{\underline{209.44}}$$

(b) $\nabla \bullet \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho^2 z^2) = 4z^2$

$$\int \nabla \bullet \mathbf{D} dv = \iiint 4z^2 \rho d\rho d\phi dz = 4\int_{-1}^{1} z^2 dz \int_0^5 \rho d\rho \int_0^{2\pi} d\phi$$

$$= 4x\frac{z^3}{3}\Big|_{-1}^{1} \frac{\rho^2}{2}\Big|_0^5 (2\pi) = \frac{200\pi}{3} = \underline{\underline{209.44}}$$

Prob. 3.24

$$\int_S \mathbf{H} \bullet d\mathbf{S} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} 10\cos\theta r^2 \sin\theta d\theta d\phi \Big|_{r=1}$$

$$= 10(1)^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta\cos\theta d\theta = 10(2\pi)\int_0^{\pi/2} \sin\theta d(\sin\theta)$$

$$= 20\pi\left(\frac{\sin^2\theta}{2}\right)\Big|_0^{\pi/2} = 10\pi = \underline{\underline{31.416}}$$

Prob. 3.25

(a)

$$\oint \mathbf{A} \bullet d\mathbf{S} = \int_v \nabla \bullet \mathbf{A} dv, \quad \nabla \bullet \mathbf{A} = y + z + x$$

$$\oint \mathbf{A} \bullet d\mathbf{S} = \int_0^1\int_0^1\int_0^1 (x+y+z)dxdydz$$

$$= 3\int_0^1 \lambda d\lambda \int_0^1 d\mu \int_0^1 d\tau = 3(\frac{\lambda^2}{2}\Big|_0^1)(1)(1)$$

$$= \underline{\underline{1.5}}$$

(b)

$\nabla \bullet \mathbf{A} = 0.$ Hence, $\oint \mathbf{A} \bullet d\mathbf{S} = \underline{\underline{0}}$

Prob. 3.26

$$\oint_S \mathbf{H} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{H} dv$$

$$\oint_S \mathbf{H} \cdot d\mathbf{S} = -\iint 2xydydz\Big|_{x=0} + \iint 2xydydz\Big|_{x=1} - \iint (x^2+z^2)dxdz\Big|_{y=1}$$

$$+ \iint (x^2+z^2)dxdz\Big|_{y=2} - \iint 2yzdxdy\Big|_{z=-1} + \iint 2yzdxdy\Big|_{z=3}$$

$$= 0 + 2\int_1^2 ydy \int_{-1}^{3} dz + 2\int_0^1 dx \int_1^2 ydy + 6\int_0^1 dx \int_1^2 ydy$$

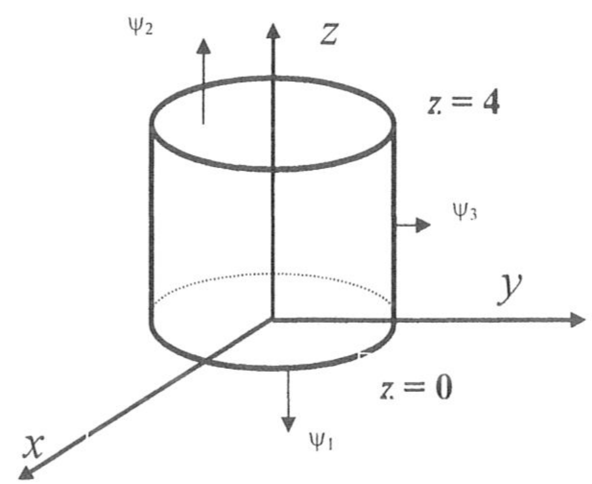
$$= 12 + 3 + 9 = \underline{\underline{24}}$$

$$\nabla \cdot \mathbf{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 2y + 0 + 2y = 4y$$

$$\int_V \nabla \cdot \boldsymbol{H} dv = \iiint 4y dx dy dz = 4\int_0^1 dx \int_1^2 y dy \int_{-1}^3 dz$$

$$= 4(1)\frac{y^2}{2}\Big|_1^2 (3+1) = \underline{\underline{24}}$$

Prob. 3.27



Side 1:

$$\psi = \oint_S \boldsymbol{B} \bullet ds = \psi_1 + \psi_2 + \psi_3$$

$$= 0 + \int_{\phi=0}^{2\pi}\int_{\rho=0}^{3} 10z\,\rho d\phi d\rho\Big|_{z=4} + \int_{z=0}^{4}\int_{\phi=0}^{2\pi} 5\,\rho\rho d\phi dz\Big|_{\rho=3}$$

$$= 10(4)(2\pi)(9/2) + 5(9)(2\pi)4 = \underline{\underline{2261.95}}$$

Side 2:

$$\psi = \int_v \nabla \bullet \boldsymbol{B} dv, \qquad \nabla \bullet \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(5\rho^2) + 0 + 10 = 10 + 10 = 20$$

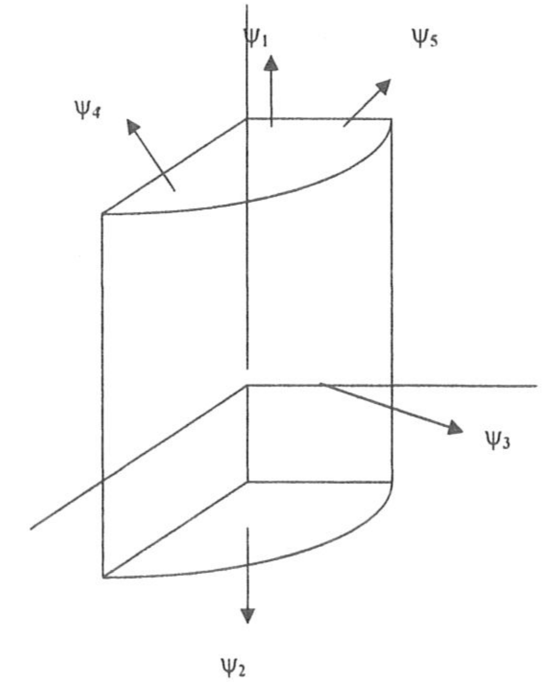
$$\psi = \iiint 20 dv = 20\int_{\phi=0}^{2\pi}\int_{z=0}^{4}\int_{\rho=0}^{3} \rho d\phi d\rho dz = 20(2\pi)(4)\left(\frac{\rho^2}{2}\Big|_0^3\right)$$

$$= 720\pi = \underline{\underline{2261.95}}$$

Prob. 3.28

$$\psi = \oint_S \boldsymbol{F} \bullet d\boldsymbol{S} = \int_v \nabla \bullet \boldsymbol{F} dv$$

Side 1:



Let $\psi = \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5$

$$= \int_{\phi=0}^{\pi/2}\int_{\rho=0}^{1} z\,\rho d\rho d\phi\Big|_{z=2} - \int_{\phi=0}^{\pi/2}\int_{\rho=0}^{1} z\,\rho d\rho d\phi\Big|_{z=-1} + \int_{\phi=0}^{\pi/2}\int_{z=-1}^{2} 10\rho\sin\phi\rho d\phi dz\Big|_{\rho=1}$$

$$- \int_{\rho=0}^{1}\int_{z=-1}^{2} 4\rho\cos\phi d\rho dz\Big|_{\phi=0} + \int_{\rho=0}^{1}\int_{z=-1}^{2} 4\rho\cos\phi d\rho dz\Big|_{\phi=\pi/2}$$

The last term is zero since $\cos 90^\circ = 0$.

$$\psi = 2\left(\frac{\pi}{2}\right)\left(\frac{\rho^2}{2}\Big|_{\rho=0}^{1}\right) - (-1)\left(\frac{\pi}{2}\right)\left(\frac{\rho^2}{2}\Big|_{\rho=0}^{1}\right) + 10(1)(3)\left(-\cos\phi\Big|_{\phi=0}^{\pi/2}\right) - 4(3)\left(\frac{\rho^2}{2}\Big|_{\rho=0}^{1}\right)$$

$$= \left(\frac{\pi}{2}\right) + \left(\frac{\pi}{4}\right) + 30 - 6 = 24 + \left(\frac{3\pi}{4}\right) = \underline{\underline{26.356}}$$

<u>Side 2:</u>

$$\nabla \bullet \boldsymbol{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{dz} = 20\sin\phi - 4\sin\phi + 1$$

$$= 16\sin\phi + 1$$

$$\psi = \iiint (16\sin\phi + 1)\rho d\phi d\rho dz = 16\int_{\rho=0}^{1}\rho d\rho \int_0^{\pi/2}\sin\phi d\phi \int_{-1}^{2} dz + \int_0^1 \rho d\rho \int_0^{\pi/2} d\phi \int_{-1}^{2} dz$$

$$= 16(1/2)(3) + \left(\frac{\rho^2}{2}\Big|_0^1\right)(\pi/2)(3) = 24 + \frac{3\pi}{4} = \underline{\underline{26.356}}$$

Prob. 3.29

$$\nabla \bullet \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi)$$

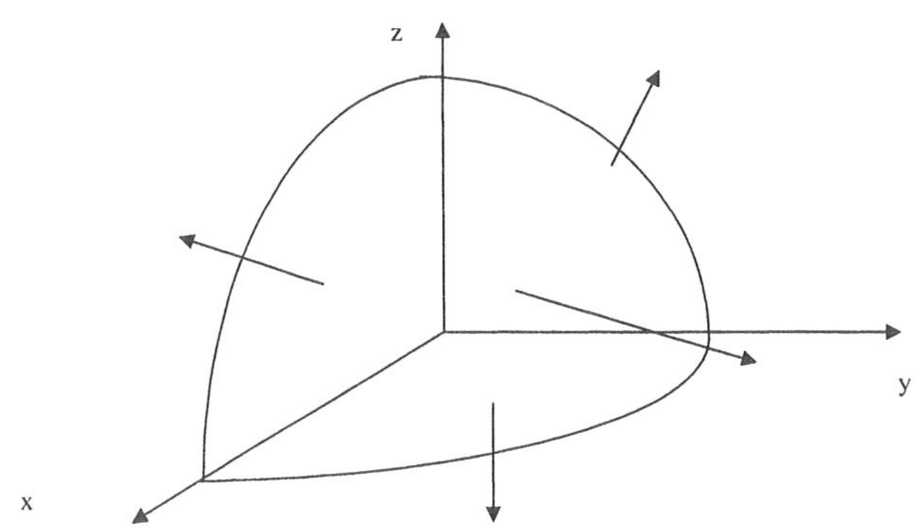
$$= 4r + 2\cos\theta\cos\phi$$

$$\int \nabla \bullet \boldsymbol{A} dv = \iiint 4r^3 \sin\theta d\theta d\phi dr + \iiint 2r^2 \sin\theta\cos\theta\cos\phi d\theta d\phi dr$$

$$= 4\frac{r^4}{4}\Big|_0^3 (-\cos\theta)\Big|_0^{\pi/2}(\frac{\pi}{2}) + \frac{2r^3}{3}\Big|_0^3 (-\frac{\cos^2\theta}{2})\Big|_0^{\pi/2} \sin\phi\Big|_0^{\frac{\pi}{2}}$$

$$= 81(1)(\frac{\pi}{2}) + 18(0 + \frac{1}{2})(1 - 0)$$

$$= \frac{81\pi}{2} + 9 = \underline{\underline{136.23}}$$



$$\int \boldsymbol{A} \bullet d\boldsymbol{S} = [\iint_{\phi=0} + \iint_{\phi=\pi/2} + \iint_{r=3} + \iint_{\theta=\pi/2}]\boldsymbol{A} \bullet d\boldsymbol{S}$$

Since $\boldsymbol{A}$ has no $\phi-$component, the first two integrals on the right hand side vanish.

$$\int \boldsymbol{A} \bullet d\boldsymbol{S} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} r^4 \sin\theta \, d\theta \, d\phi\Big|_{r=3} + \int_{r=0}^{3}\int_{\phi=0}^{\pi/2} r^2\sin^2\theta\cos\phi dr d\phi\Big|_{\theta=\pi/2}$$

$$= 81\,(\frac{\pi}{2})\,(-\cos\theta)\Big|_0^{\pi/2} + 9(1)\sin\phi\Big|_0^{\pi/2}$$

$$= \frac{81\pi}{2} + 9 \quad = \underline{\underline{136.23}}$$

Prob. 3.30

Let $\psi = \oint \boldsymbol{F} \bullet d\boldsymbol{S} = \psi_t + \psi_b + \psi_o + \psi_i$

where ψ_t, ψ_b, ψ_o, ψ_i are the fluxes through the top surface, bottom surface, outer surface ($\rho = 3$), and inner surface respectively.

For the top surface, $d\boldsymbol{S} = \rho d\phi d\rho \boldsymbol{a}_z$, $\quad z = 5$;

$\boldsymbol{F} \bullet d\boldsymbol{S} = \rho^2 \, z \, d\phi \, dz$. Hence:

$$\psi_t = \int_{\rho=2}^{3}\int_{\phi=0}^{2\pi} \rho^2 \, z \, d\phi \, dz\Big|_{z=5} = \frac{190\ \pi}{3}$$

For the bottom surface, $\quad z = 0$, $d\boldsymbol{S} = \rho d\phi d\rho(-\boldsymbol{a}_z)$

$\boldsymbol{F} \bullet d\boldsymbol{S} = -\rho^2 z d\phi \, d\rho = 0$. Hence, $\psi_b = 0$.

For the outer curved surface, $\rho = 3$, $\quad d\boldsymbol{S} = \rho d\phi dz \, \mathbf{a}_\rho$

$\boldsymbol{F} \bullet d\boldsymbol{S} = \rho^2 \sin\phi \, d\phi \, dz$. Hence,

$$\psi_o = \int_{z=0}^{5} dz \, \rho^3 \int_{\phi=0}^{2\pi} \sin\phi \, d\phi\Big|_{\rho=3} = 0$$

For the inner curved surface, $\rho = 2$, $\quad d\,\boldsymbol{S} = \rho d\phi dz(-\boldsymbol{a}_\rho)$

$\boldsymbol{F} \bullet d\boldsymbol{S} = -\rho^3 \sin\phi \, d\phi \, dz$. Hence,

$$\psi_a = -\int_{z=0}^{5} dz \, \rho^3 \int_{\phi=0}^{2\pi} \sin\phi \, d\phi\Big|_{\rho=2} = 0$$

$$\psi = \frac{190\pi}{3} + 0 + 0 + 0 = \underline{\underline{\frac{190\ \pi}{3}}}$$

$$\psi = \oint F \bullet dS = \int \nabla \bullet F dV$$

$$\nabla \bullet F = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^3 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(z\cos\phi) + \rho$$

$$= 3\rho\sin\phi - \frac{z}{\rho}\sin\phi + \rho$$

$$\int_v \nabla \bullet F dv = \iiint (3\rho\sin\phi - \frac{z}{\rho}\sin\phi + \rho)\ \rho\, d\phi\, d\rho\, dz$$

$$= 0 + 0 + \int_0^5 dz \int_0^{2\pi} d\phi \int_2^3 \rho^2 d\rho$$

$$= \underline{\underline{\frac{190\pi}{3}}}$$

Prob. 3.31

(a)

$$\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 & -xz \end{vmatrix} = z a_y - x a_z$$

(b)

$$\nabla \times B = \left(\frac{1}{\rho} 2\rho z 2\sin\phi\cos\phi - 0\right) a_\rho + (2\rho z - 2z\sin^2\phi) a_\phi + \frac{1}{\rho}\left(2\rho\sin^2\phi - 0\right) a$$

$$= 4z\sin\phi\cos\phi a_\rho + 2(\rho z - z\sin^2\phi) a_\phi + 2\sin^2\phi a_z$$

$$= \underline{\underline{2z\sin 2\phi a_\rho + 2z(\rho - \sin^2\phi) a_\phi + 2\sin^2\phi a_z}}$$

(c)

$$\nabla \times C = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(r\cos^2\theta\sin\theta)\right] a_r - \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\cos^2\theta\right] a_\theta$$

$$= \frac{r}{r\sin\theta}\left[(2\cos\theta)(-\sin\theta)\sin\theta + \cos\theta(\cos^2\theta)\right] a_r - \frac{\cos^2\theta}{r}(2r) a_\theta$$

$$= \underline{\underline{\frac{(\cos^3\theta - 2\sin^2\theta\cos\theta)}{\sin\theta} a_r - 2\cos^2\theta a_\theta}}$$

Prob. 3.32

(a)

$$\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & y^2 z & -2xz \end{vmatrix} = \underline{\underline{-y^2 a_x + 2z a_y - x^2 a_z}}$$

$$\nabla \bullet \nabla \times A = \underline{\underline{0}}$$

(b)

$$\nabla \times A = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) a_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) a_\phi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\rho)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right) a_z$$

$$= (0-0) a_\rho + (\rho^2 - 3z^2) a_\phi + \frac{1}{\rho}(4\rho^3 - 0) a_z$$

$$= \underline{\underline{(\rho^2 - 3z^2) a_\phi + 4\rho^2 a_z}}$$

$$\nabla \bullet \nabla \times A = \underline{\underline{0}}$$

(c)

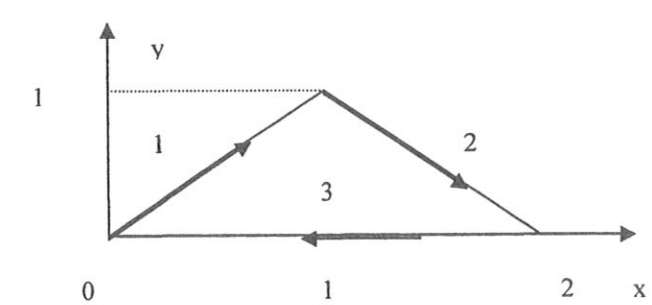
$$\nabla \times A = \frac{1}{r\sin\theta}\left[0 - \frac{\sin\phi}{r^2}\right] a_r + \frac{1}{r}\left[\frac{-1}{\sin\theta}\frac{\cos\phi}{r^2} - 0\right] a_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{\cos\phi)}{r}\right) - 0\right] a_\phi$$

$$= -\frac{\sin\phi}{r^3\sin\theta} a_r + \frac{\cos\phi}{r^3\sin\theta} a_\theta + \frac{\cos\phi}{r^3} a_\phi$$

$$\nabla \bullet \nabla \times A = \frac{-\sin\phi}{r^4\sin\theta} + 0 + \frac{\sin\phi}{r^4\sin\theta} = 0$$

$$\nabla \bullet \nabla \times A = \underline{\underline{0}}$$

Prob. 3.33



(a)

$$\oint_L \boldsymbol{F}\bullet d\boldsymbol{l} = \left(\int_1 + \int_2 + \int_3\right)\boldsymbol{F}\bullet d\boldsymbol{l}$$

For 1, $y = x \quad dy = dx, d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y.$

$$\int_1 \boldsymbol{F}\bullet d\boldsymbol{l} = \int_0^1 x^3dx - xdx = -\frac{1}{4}$$

For 2, $y = -x+2, dy = -dx, d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y.$

$$\int_2 \boldsymbol{F}\bullet d\boldsymbol{l} = \int_1^2(-x^3+2x^2-x+2)dx = \frac{17}{12}$$

For 3,

$$\int_3 \boldsymbol{F}\bullet d\boldsymbol{l} = \int_2^0 x^2ydx\Big|_{y=0} = 0$$

$$\oint_L \boldsymbol{F}\bullet d\boldsymbol{l} = -\frac{1}{4} + \frac{17}{12} + 0 = \underline{\underline{\frac{7}{6}}}$$

(b)

$\nabla\times\boldsymbol{F} = -x^2\boldsymbol{a}_z$; $\quad d\boldsymbol{S} = dxdy(-\boldsymbol{a}_z)$

$$\int(\nabla\times\boldsymbol{F})\bullet dS = -\iint(-x^2)dxdy = \int_0^1\int_0^x x^2dydx + \int_1^2\int_{y=0}^{-x+2} x^2dydx$$

$$= \int_0^1 x^2y\Big|_0^x dx + \int_1^2 x^2y\Big|_0^{-x+2}dx = \frac{x}{4}\Big|_0^1 + \int_1^2 x^2(-x+2)dx = \underline{\underline{\frac{7}{6}}}$$

(c) <u>Yes</u>

Prob.3.34

$$\oint \boldsymbol{A}\bullet d\boldsymbol{l} = \int_{\rho=2}^{1}\rho\sin\phi d\rho\Big|_{\phi=0} + \int_{\phi=0}^{\pi/2}\rho^2\rho d\phi\Big|_{\rho=1} + \int_{\rho=1}^{2}\rho\sin\phi d\rho\Big|_{\phi=90^\circ} + \int_{\phi=\pi/2}^{0}\rho^3 d\phi\Big|_{\rho=2}$$

$$= \frac{\pi}{2} + \frac{1}{2}(4-1) + 8\left(-\frac{\pi}{2}\right) = \underline{\underline{-9.4956}}$$

Prob. 3.35

We need coordinate transformation to get F_ϕ.

$$\begin{bmatrix}F_\rho\\F_\phi\\F_z\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2xy\\ y\\ 0\end{bmatrix}$$

$$F_\phi = -2xy\sin\phi + y\cos\phi$$

But $x = \rho\cos\phi, \quad y = \rho\sin\phi$

$$F_\phi = -2\rho^2\cos\phi\sin^2\phi + \rho\sin\phi\cos\phi$$

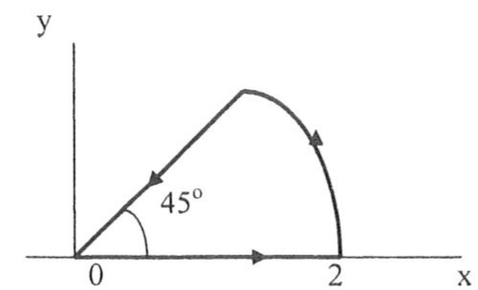
$$\oint_L \boldsymbol{F}\cdot dl = \int F_y dy\Big|_{x=2} + \int F_x dx\Big|_{y=2} + \int F_\phi\rho d\phi\Big|_{\rho=2}$$

$$= \frac{y^2}{2}\Big|_0^2 + x^2y\Big|_2^0 + \int_{\pi/2}^{0}(-16\cos\phi\sin^2\phi + 4\sin\phi\cos\phi)d\phi$$

$$= 2 - 8 - 16\int_{\pi/2}^{0}\sin^2\phi d(\sin\phi) + 4\int_{\pi/2}^{0}\sin\phi d(\sin\phi)$$

$$= -6 + 16/3 - 4/2 = \underline{\underline{-2.67}}$$

Prob. 3.36



$$\oint \boldsymbol{F}\bullet dl = \int_0^2 2\rho z d\rho\Big|_{z=1} + \int_0^{\pi/4} 3z\sin\phi\,\rho d\phi\Big|_{\rho=2, z=1} + \int_2^0 2\rho z d\rho\Big|_{z=1}$$

$$= \rho^2\Big|_0^2 + (-6\cos\phi)\Big|_0^{\pi/4} + \rho^2\Big|_2^0 = (4-0) + 6(-\cos\pi/4 + 1) + (0-4) = \underline{\underline{1.757}}$$

$$\nabla x \mathbf{F} = \frac{1}{\rho}\left[3z\sin\phi - 0\right]\mathbf{a}_z + ...$$

$$\int(\nabla x\mathbf{F})\cdot d\mathbf{S} = \int_{\rho=0}^{2}\int_{\phi=0}^{\pi/4}\frac{3z}{\rho}\sin\phi\,\rho d\phi d\rho\Big|_{z=} \quad = 3(2)(-\cos\phi)\Big|_0^{\pi/4}$$

$$= 6(-\cos\pi + 1) = \underline{\underline{1.757}}$$

Prob. 3.37

$$\nabla\cdot A = 8xe^{-y} + 8xe^{-y} = 16xe^{-y}$$
$$\nabla(\nabla\cdot\mathbf{A}) = 16e^{-y}\mathbf{a}_x - 16xe^{-y}\mathbf{a}_y$$

$$\nabla x\nabla(\nabla\cdot\mathbf{A}) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16e^{-y} & -16xe^{-y} & 0 \end{vmatrix} = (-16e^{-y} + 16e^{-y})\mathbf{a}_z = \underline{\underline{0}}$$

Should be expected since $\nabla x\nabla V = 0$.

Prob. 3.38

(a) $\nabla V = \underline{\underline{-\frac{\sin\theta\cos\phi}{r^2}\mathbf{a}_r + \frac{\cos\theta\cos\phi}{r^2}\mathbf{a}_\theta - \frac{\sin\phi}{r^2}\mathbf{a}_\phi}}$

(b) $\nabla x\nabla V = \underline{\underline{0}}$

(c)

$$\nabla\cdot\nabla V = \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(-\sin\theta\cos\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\cos\theta\cos\phi}{r}) + \frac{1}{r^2\sin^2\theta}(-\frac{\sin\theta\cos\phi}{r})$$

$$= 0 + \frac{\cos\phi}{r^3\sin\theta}(1 - 2\sin^2\theta) - \frac{\cos\phi}{r^3\sin\theta}$$

$$= \underline{\underline{-\frac{2\sin\theta\cos\phi}{r^3}}}$$

Prob.3.39

$$Q = \frac{r}{r\sin\theta}r\sin\theta[(\cos\phi - \sin\phi)\mathbf{a}_x + (\cos\phi + \sin\phi)\mathbf{a}_y]$$

$$= r(\cos\phi - \sin\phi)\mathbf{a}_x + r(\cos\phi + \sin\phi)\mathbf{a}_y$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}\begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix}$$

$$\mathbf{Q} = r\sin\theta\,\mathbf{a}_r + r\cos\theta\,\mathbf{a}_\theta + r\,\mathbf{a}_\phi$$

(a)

$$dl = \rho\, d\phi\, \mathbf{a}_\phi, \quad \rho = r\sin 30° = 2(\frac{1}{2}) = 1$$

$$z = r\cos 30° = \sqrt{3}$$

$$Q_\phi = r = \sqrt{\rho^2 + z^2}$$

$$\oint \mathbf{Q}\bullet dl = \int_0^{2\pi}\sqrt{\rho^2 + z^2}\,\rho\, d\phi = 2(1)(2\pi) = \underline{\underline{4\pi}}$$

(b)

$$\nabla\times\mathbf{Q} = \cot\theta\,\mathbf{a}_r - 2\mathbf{a}_\theta + \cos\theta\,\mathbf{a}_\phi$$

For S_1, $d\mathbf{S} = r^2\sin\theta\, d\theta\, d\phi\, \mathbf{a}_r$

$$\int_{S_1}(\nabla\times\mathbf{Q})\bullet d\mathbf{S} = \int r^2\sin\theta\cot\theta\, d\theta\, d\phi\Big|_{r=2}$$

$$= 4\int_0^{2\pi}d\phi\int_0^{30°}\cos\theta\, d\theta = \underline{\underline{4\pi}}$$

(c)

For S_2, $d\mathbf{S} = r\sin\theta d\theta dr\,\mathbf{a}_\theta$

$$\int_{S_2}(\nabla\times\mathbf{Q})\bullet d\mathbf{S} = -2\int r\sin\theta\, d\phi\, dr\Big|_{\theta=30°}$$

$$= -2\sin 30\int_0^2 r dr\int_0^{2\pi}d\phi$$

$$= \underline{\underline{-4\pi}}$$

(d)

For S_1, $d\mathbf{S} = r^2 \sin\theta \, d\phi \, d\theta \, \mathbf{a}_r$

$$\int_{S_1} \mathbf{Q} \bullet d\mathbf{S} = r^3 \int \sin^2\theta \, d\theta \, d\phi \Big|_{r=2}$$

$$= 8 \int_0^{2\pi} d\phi \int_0^{30^\circ} \sin^2\theta \, d\theta$$

$$= 4\pi[\frac{\pi}{3} - \frac{\sqrt{3}}{2}] = \underline{\underline{2.2767}}$$

(e)

For S_2, $d\mathbf{S} = r \sin\theta \, d\phi \, dr \, \mathbf{a}_\theta$

$$\int_{S_2} \mathbf{Q} \bullet d\mathbf{S} = \int r^2 \sin\theta \cos\theta \, d\phi \, dr \Big|_{\theta=30^\circ}$$

$$= \frac{4\pi\sqrt{3}}{3} = \underline{\underline{7.2552}}$$

(f)

$$\nabla \bullet \boldsymbol{Q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3 \sin\theta) + \frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta) + 0$$
$$= 2\sin\theta + \cos\theta\cot\theta$$

$$\int \nabla \bullet \mathbf{Q} dv = \int (2\sin\theta + \cos\theta\cot\theta) r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{r^3}{3}\Bigg|_0^2 (2\pi) \int_0^{30} (1+\sin^2\theta) d\theta$$

$$= \frac{4\pi}{3}(\pi - \frac{\sqrt{3}}{2}) = \underline{\underline{9.532}}$$

$$Check: \int \nabla \bullet \mathbf{Q} dv = (\int_{S_1} + \int_{S_2} \mathbf{Q} \bullet d\mathbf{S}$$

$$= 4\pi[\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}]$$

$$= \frac{4\pi}{3}[\pi - \frac{\sqrt{3}}{2}] \quad \text{(It checks!)}$$

Prob. 3.40

Since $\mathbf{u} = \omega \times \mathbf{r}$, $\nabla \times \mathbf{u} = \nabla \times (\omega \times \mathbf{r})$. From Appendix A.10,

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \bullet \mathbf{B}) - \mathbf{B}(\nabla \bullet \mathbf{A}) + (\mathbf{B} \bullet \nabla)\mathbf{A} - (\mathbf{A} \bullet \nabla)\mathbf{B}$$

$$\nabla \times \boldsymbol{u} = \nabla \times (\omega \times \boldsymbol{r})$$

$$\nabla \times (\omega \times \boldsymbol{r}) = \omega(\nabla \bullet \boldsymbol{r}) - \boldsymbol{r}(\nabla \bullet \omega) + (\boldsymbol{r} \bullet \nabla)\omega - (\omega \bullet \nabla)\boldsymbol{r}$$
$$= \omega(3) - \omega = 2\,\omega$$

or $\omega = \frac{1}{2}\nabla \times \boldsymbol{u}$.

Alternatively, let $x = r\cos\omega t$, $y = r\sin\omega t$

$$u = \frac{\partial x}{\partial t}\boldsymbol{a}_x + \frac{\partial y}{\partial t}\boldsymbol{a}_y$$

$$= -\omega r \sin\omega t \, \boldsymbol{a}_x + \omega r \cos\omega t \, \boldsymbol{a}_y$$

$$= -\omega y \boldsymbol{a}_x + \omega x \boldsymbol{a}_y$$

$$\nabla \times \boldsymbol{u} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega \boldsymbol{a}_z = 2\omega$$

i.e., $\omega = \frac{1}{2}\nabla \times \boldsymbol{u}$

Note that we have used the fact that $\nabla \bullet \omega = 0$, $(\boldsymbol{r} \bullet \nabla)\omega = 0$, $(\omega \bullet \nabla)\boldsymbol{r} = \omega$

Prob. 3.41

(a)

$$\nabla \bullet \boldsymbol{H} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\rho) + \frac{1}{\rho}\frac{\partial H_\phi}{\partial\phi} + \frac{\partial H_z}{\partial z} = \frac{1}{\rho}6\rho z + 0 + 0 = \underline{\underline{6z}}$$

$$\nabla \times \boldsymbol{H} = \left[\frac{1}{\rho}\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right]\boldsymbol{a}_\rho + \left[\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial\rho}\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho H_\phi) - \frac{\partial H_\rho}{\partial\phi}\right]\boldsymbol{a}_z$$

$$= \frac{1}{\rho}4\rho\sin\phi \boldsymbol{a}_\rho + (3\rho + 4\cos\phi)\boldsymbol{a}_\phi + 0$$

$$= \underline{\underline{4\sin\phi \boldsymbol{a}_\rho + (3\rho + 4\cos\phi)\boldsymbol{a}_\phi}}$$

(b)

$$\nabla\bullet \boldsymbol{G} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 G_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(G_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial G_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^3) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta) + 0$$

$$= 3 + \frac{1}{\sin\theta}2\sin\theta\cos\theta = \underline{\underline{3+2\cos\theta}}$$

$$\nabla\times \boldsymbol{G} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(G_\phi\sin\theta) - \frac{\partial G_\theta}{\partial\phi}\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial G_r}{\partial\phi} - \frac{\partial}{\partial r}(rG_\phi)\right]\boldsymbol{a}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rG_\theta) - \frac{\partial G_r}{\partial\theta}\right]\boldsymbol{a}_\phi$$

$$= 0 + 0 + \frac{1}{r}\frac{\partial}{\partial r}(r^2\sin\theta)\boldsymbol{a}_\phi$$

$$= \underline{\underline{2\sin\theta\boldsymbol{a}_\phi}}$$

Prob. 3.42

(a)

$$\nabla\bullet(V\nabla V) = \nabla\bullet\left(V\frac{\partial V}{\partial x}\boldsymbol{a}_x + V\frac{\partial V}{\partial y}\boldsymbol{a}_y + V\frac{\partial V}{\partial z}\boldsymbol{a}_z\right)$$

$$= \frac{\partial}{\partial x}\left(V\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(V\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(V\frac{\partial V}{\partial z}\right)$$

$$= V\frac{\partial^2 V}{\partial x^2} + V\frac{\partial^2 V}{\partial y^2} + V\frac{\partial^2 V}{\partial z^2} + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2$$

$$= V\nabla^2 V + |\nabla V|^2$$

(b)

$$\nabla\times V\boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ VA_x & VA_y & VA_z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(VA_z) - \frac{\partial}{\partial z}(VA_y)\right]\boldsymbol{a}_x + \left[\frac{\partial}{\partial z}(VA_x) - \frac{\partial}{\partial x}(VA_z)\right]\boldsymbol{a}_y + \left[\frac{\partial}{\partial x}(VA_y) - \frac{\partial}{\partial y}(VA_x)\right]\boldsymbol{a}_z$$

$$= \left[A_z\frac{\partial V}{\partial y} + V\frac{\partial A_z}{\partial y} - A_y\frac{\partial V}{\partial z} - V\frac{\partial A_y}{\partial z}\right]\boldsymbol{a}_x$$

$$+ \left[A_x\frac{\partial V}{\partial z} + V\frac{\partial A_x}{\partial z} - A_z\frac{\partial V}{\partial x} - V\frac{\partial A_z}{\partial x}\right]\boldsymbol{a}_y$$

$$+ \left[A_y\frac{\partial V}{\partial x} + V\frac{\partial A_y}{\partial x} - A_x\frac{\partial V}{\partial y} - V\frac{\partial A_x}{\partial y}\right]\boldsymbol{a}_z$$

$$\nabla\times V\boldsymbol{A} = V\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{a}_z\right]$$

$$+ \left(A_z\frac{\partial V}{\partial y} - A_y\frac{\partial V}{\partial z}\right)\boldsymbol{a}_x + \left(A_x\frac{\partial V}{\partial z} - A_z\frac{\partial V}{\partial x}\right)\boldsymbol{a}_y + \left(A_y\frac{\partial V}{\partial x} - A_x\frac{\partial V}{\partial y}\right)\boldsymbol{a}_z$$

$$= V\nabla\times\boldsymbol{A} + \nabla V\times\boldsymbol{A}$$

Prob. 3.43

Method 1:

$$\nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2z^2-5y) & 6xz^2 & (2x+y) \end{vmatrix} = (1-12xz)\boldsymbol{a}_x + (4z-2)\boldsymbol{a}_y + (6z^2+5)\boldsymbol{a}_z$$

$$\nabla\times\nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (1-12xz) & (4z-2) & (6z^2+5) \end{vmatrix} = -4\boldsymbol{a}_x - 12x\boldsymbol{a}_y$$

At (1,2, -3), $x = 1$,

$$\nabla\times\nabla\times\boldsymbol{F} = \underline{\underline{-4\boldsymbol{a}_x - 12\boldsymbol{a}_y}}$$

Method 2:

$$\nabla\times\nabla\times \boldsymbol{F} = \nabla(\nabla\bullet \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$$

$$\nabla\bullet \boldsymbol{F} = 0$$

$$\nabla^2 \boldsymbol{F} = \nabla^2 F_x \boldsymbol{a}_x + \nabla^2 F_y \boldsymbol{a}_y + \nabla^2 F_z \boldsymbol{a}_z = 4\boldsymbol{a}_x + 12x\boldsymbol{a}_y$$

$$\nabla\times\nabla\times \boldsymbol{F} = 0 - (4\boldsymbol{a}_x + 12x\boldsymbol{a}_y) = \underline{\underline{-4\boldsymbol{a}_x - 12x\boldsymbol{a}_y}}$$

Prob. 3.44

(a)

$$V_1 = x^3 + y^3 + z^3$$

$$\nabla^2 V_1 = \frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(3y^2) + \frac{\partial}{\partial x}(3z^2)$$

$$= 6x + 6y + 6z = \underline{\underline{6(x+y+z)}}$$

(b)

$$V_2 = \rho z^2 \sin 2\phi$$

$$\nabla^2 V_2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho z^2 \sin 2\phi) - \frac{4z^2}{\rho}\sin 2\phi + \frac{\partial}{\partial z}(2\rho z \sin 2\phi)$$

$$= \frac{z^2}{\rho}\sin 2\phi - \frac{4z^2}{\rho}\sin 2\phi + 2\rho \sin 2\phi$$

$$= \underline{\underline{\left(\frac{-3z^2}{\rho} + 2\rho\right)\sin 2\phi}}$$

(c)

$$V_3 = r^2(1 + \cos\theta\sin\phi)$$

$$\nabla^2 V_3 = \frac{1}{r^2}\frac{\partial}{\partial r}[2r^3(1+\cos\theta\sin\phi)]$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(-\sin^2\theta\sin\phi)r^2 + \frac{1}{r^2\sin^2\theta}r^2(-\cos\theta\sin\phi)$$

$$= 6(1+\cos\theta\sin\phi) - \frac{2\sin\theta}{\sin\theta}\cos\theta\sin\phi - \frac{\cos\theta\sin\phi}{\sin^2\theta}$$

$$= \underline{\underline{6 + 4\cos\theta\sin\phi - \frac{\cos\theta\sin\phi}{\sin^2\theta}}}$$

Prob. 3.45

(a)

$$U = x^3 y^2 e^{xz}$$

$$\nabla^2 U = \frac{\partial}{\partial x}(3x^2y^2e^{xz} + x^3y^2ze^{xz}) + \frac{\partial}{\partial y}(2x^3ye^{xz}) + \frac{\partial}{\partial z}(x^4y^2e^{xz})$$

$$= 6xy^2e^{xz} + 3x^2yze^{xz} + 3x^2y^2ze^{xz} + x^3y^2z^2e^{xz} + 2x^3e^{xz} + x^5y^2e^{xz}$$

$$= \underline{\underline{e^{xz}(6xy^2 + 3x^2y^2z + 3x^2y^2z + x^3y^2z^2 + 2x^3 + x^5y^2)}}$$

At $(1,-1,1)$,

$$\nabla^2 U = e^1(6+3+3+1+2+1) = 16e = \underline{\underline{43.493}}$$

(b)

$$V = \rho^2 z(\cos\phi + \sin\phi)$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}[2\rho^2 z(\cos\phi+\sin\phi)] - z(\cos\phi+\sin\phi) + 0$$

$$= 4z(\cos\phi+\sin\phi) - z(\cos\phi+\sin\phi)$$

$$= \underline{\underline{3z(\cos\phi+\sin\phi)}}$$

At $(5, \frac{\pi}{6}, -2)$, $\nabla^2 V = -6(0.866+0.5) = \underline{\underline{-8.196}}$

(c)

$$W = e^{-r}\sin\theta\cos\phi$$

$$\nabla^2 W = \frac{1}{r^2}\frac{\partial}{\partial r}(-r^2e^{-r}\sin\theta\cos\phi) + \frac{e^{-r}}{r^2\sin\theta}\cos\phi\frac{\partial}{\partial\theta}(\sin\theta\cos\theta)$$

$$- \frac{e^{-r}\sin\theta\cos\phi}{r^2\sin^2\theta}$$

$$= \frac{1}{r^2}(-2re^{-r}\sin\theta\cos\phi) + e^{-r}\sin\theta\cos\phi$$

$$+ \frac{e^{-r}\cos\phi}{r^2\sin\theta}(1-2\sin^2\theta) - \frac{e^{-r}\cos\phi}{r^2\sin\theta}$$

$$\underline{\underline{\nabla^2 W = e^{-r}\sin\theta\cos\phi\left(1 - \frac{2}{r} - \frac{2}{r^2}\right)}}$$

At $(1, 60°, 30°)$,

$$\nabla^2 W = e^{-1}\sin 60\cos 30(1-2-2) = -2.25e^{-1} = \underline{\underline{-0.8277}}$$

Prob. 3.46

(a) Let $V = \ln r = \ln\sqrt{x^2+y^2+z^2}$

$$\frac{\partial V}{\partial x} = \frac{1}{r}\,\frac{1}{2}(2x)\,(x^2+y^2+z^2)^{-1/2} = \frac{x}{r^2}$$

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z}{r^2} = \underline{\underline{\frac{\mathbf{r}}{r^2}}}$$

(b) Let $\nabla V = \mathbf{A} = \frac{\mathbf{r}}{r^2} = \frac{1}{r}\mathbf{a}_r$ in spherical coordinates.

$$\nabla^2(\ln r) = \nabla\bullet\nabla(\ln r) = \nabla\bullet\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r)$$

$$= \underline{\underline{\frac{1}{r^2}}}$$

Prob. 3.47

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

$$= y^2z^3\mathbf{a}_x + 2xyz^3\mathbf{a}_y + 3xy^2z^2\mathbf{a}_z$$

At P(1,2,3,) $x=1, y=2, z=3$

$$\nabla V = 4(27)\mathbf{a}_x + 2(2)(27)\mathbf{a}_y + 3(4)(9)\mathbf{a}_z$$

$$= \underline{\underline{108(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial}{\partial x}\left(y^2z^3\right) + \frac{\partial}{\partial y}\left(2xyz^3\right) + \frac{\partial}{\partial z}\left(3xy^2z^2\right)$$

$$= 0 + 2xz^3 + 6xy^2z$$

$$= 2xz\left(z^2 + 3y^2\right)$$

At P(1,2,3,) $x=1, y=2, z=3$.

$$\nabla^2 V = 2(1)(3)(9+3\times 4) = 6(9+12)$$

$$\underline{\underline{=126}}$$

Prob.3.48

(a)

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

$$= \underline{\underline{-\frac{10}{r^3}\cos\phi\,\mathbf{a}_r - \frac{5\sin\phi}{r^3\sin\theta}\mathbf{a}_\phi}}$$

(b)

$$\nabla\bullet\nabla V = \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(-\frac{10\cos\phi}{r^3}\right) + 0 + \frac{1}{r^2\sin^2\theta}\left(-\frac{5\cos\phi}{r^2}\right)$$

$$\underline{\underline{\nabla\bullet\nabla V = \frac{10\cos\phi}{r^4} - \frac{5\cos\phi}{r^4\sin^2\theta}}}$$

(c) $\nabla\times\nabla V = 0$, see Example 3.10.

Prob.3.49

Method 1

$$\nabla^2\mathbf{G}\Big|_\rho = \nabla^2 G_\rho - \frac{2}{\rho^2}\frac{\partial G_\phi}{\partial\phi} - \frac{G_\rho}{\rho^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(2\rho\sin\phi\right) - \frac{2\rho\sin\phi}{\rho^2} + 0 + \frac{8\rho\sin\phi}{\rho^2} - \frac{2\rho\sin\phi}{\rho^2}$$

$$= \frac{2\sin\phi}{\rho} - \frac{2\sin\phi}{\rho} + \frac{8\sin\phi}{\rho} - \frac{2\sin\phi}{\rho} = \frac{6\sin\phi}{\rho}$$

$$\nabla^2\mathbf{G}\Big|_\phi = \nabla^2 G_\phi + \frac{2}{\rho^2}\frac{\partial G_\rho}{\partial\phi} - \frac{G_\phi}{\rho^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(4\rho\cos\phi) - \frac{1}{\rho}4\rho\cos\phi + 0 + \frac{4\rho\cos\phi}{\rho^2} - \frac{4\rho\cos\phi}{\rho^2}$$

$$= \frac{4\cos\phi}{\rho} - \frac{4\cos\phi}{\rho} + \frac{4\cos\phi}{\rho} - \frac{4\cos\phi}{\rho} = 0$$

$$\nabla^2 G\Big|_z = \nabla^2 G_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho(z^2+1)\right] + 0 + \frac{\partial}{\partial z}(2z\rho)$$

$$= \frac{1}{\rho}(z^2+1) + 2\rho$$

Adding the components together gives

$$\nabla^2 G = \frac{6\sin\phi}{\rho}\boldsymbol{a}_\rho + \left[2\rho + \frac{1}{\rho}(z^2+1)\right]\boldsymbol{a}_z$$

Method 2:

$$\nabla^2 G = \nabla(\nabla \square G) - \nabla\times(\nabla\times G)$$

Let $V = \nabla \square G = \frac{1}{\rho}\frac{\partial}{\partial\rho}(2\rho^2\sin\phi) + \frac{1}{\rho}(-4\rho\sin\phi) + 2z\rho = 2z\rho$

$$\nabla(\nabla \square G) = \nabla V = 2z\boldsymbol{a}_\rho + 2\rho\boldsymbol{a}_z$$

Let $A = \nabla\times G = \left[\frac{1}{\rho}0 - 0\right]\boldsymbol{a}_\rho + \left[0-(z^2+1)\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(4\rho^2\cos\phi) - 2\rho\cos\phi\right]\boldsymbol{a}_z$

$$= -(z^2+1)\boldsymbol{a}_\phi + 6\cos\phi\,\boldsymbol{a}_z$$

$$\nabla\times\nabla\times G = \nabla\times A = \left[-\frac{6}{\rho}\sin\phi + 2z\right]\boldsymbol{a}_\rho + (0-0)\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(-\rho(z^2+1)) - 0\right]\boldsymbol{a}_z$$

$$= \left[2z - \frac{6}{\rho}\sin\phi\right]\boldsymbol{a}_\rho - \frac{1}{\rho}(z^2+1)\boldsymbol{a}_z$$

$$\nabla^2 G = \nabla V - \nabla\times A$$

$$= 2z\boldsymbol{a}_\rho + 2\rho\boldsymbol{a}_z - \left[2z - \frac{6}{\rho}\sin\phi\right]\boldsymbol{a}_\rho + \frac{1}{\rho}(z^2+1)\boldsymbol{a}_z$$

$$= \frac{6}{\rho}\sin\phi\,\boldsymbol{a}_\rho + \left[2\rho + \frac{1}{\rho}(z^2+1)\right]\boldsymbol{a}_z$$

Prob. 3.50 (a)

$$\nabla\times G = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16xy - z & 8x^2 & -x \end{vmatrix}$$

$$= 0\boldsymbol{a}_x + (-1+1)\boldsymbol{a}_y + (16x-16x)\boldsymbol{a}_z = 0$$

Thus, G is irrotational.

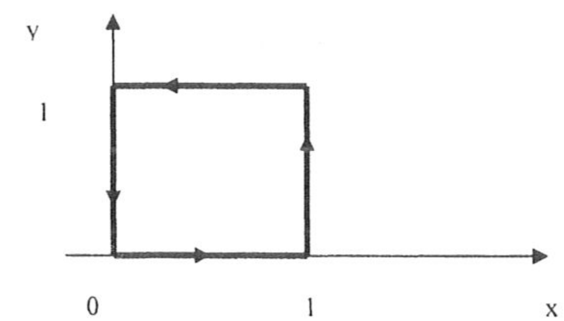
(b) Assume that ψ represents the net flux.

$$\psi = \oint G \bullet dS = \int \nabla\bullet G dv$$

$$\nabla\bullet G = 16y + 0 + 0 = 16y$$

$$\psi = \iiint 16y\,dxdydz = 16\int_0^1 dx \int_0^1 dz \int_0^1 y dy = 16(1)(1)(\frac{y^2}{2}\Big|_0^1) = \underline{\underline{8}}$$

(c)



$$\oint_L G\bullet dl = \int_{x=0}^{x=1}(16xy - z)dx\Big|_{\substack{y=0\\z=0}} + \int_{y=0}^{y=1} 8x^2 dy\Big|_{\substack{x=1\\z=0}} + \int_{x=1}^{x=0}(16xy-z)dx\Big|_{\substack{y=1\\z=0}} + \int_{y=1}^{y=0} 8x^2 dy\Big|_{\substack{x=0\\z=0}}$$

$$= 0 + 8(1)y\Big|_0^1 + 16(1)\frac{x^2}{2}\Big|_1^0 + 0$$

$$= 8 - 8 = \underline{\underline{0}}$$

This is expected since G is irrotational, i.e.

$$\oint G\bullet dl = \int(\nabla\times G)\bullet dS = 0$$

Prob. 3.51

$$\nabla\times T = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha xy + \beta z^3 & 3x^2 - \gamma z & 3xz^2 - y \end{vmatrix}$$

$$= (-1+\gamma)\boldsymbol{a}_x + (3\beta z^2 - 3z^2)\boldsymbol{a}_y + (6x - \alpha x)\boldsymbol{a}_z$$

If T is irrotational, $\nabla\times T = 0$, for all values of x,y,z. Hence,

$\underline{\alpha = 6, \beta = \gamma = 1}$

$$\nabla\bullet T = \frac{\partial}{\partial x}(\alpha xy + \beta z^3) + \frac{\partial}{\partial y}(3x^2 - \gamma z) + \frac{\partial}{\partial z}(3xz^2 - y) = 6y + 6xz$$

At (2,-1,-0),

$\nabla \bullet T = -6 + 0 = \underline{\underline{-6}}$

Prob. 3.52

$$\nabla \cdot \mathbf{F} = 0$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x-x)\boldsymbol{a}_x + (y-y)\boldsymbol{a}_y + (z-z)\boldsymbol{a}_z = \mathbf{0}$$

Hence **F** is both solenoidal and conservative.

CHAPTER 4

P. E. 4.1

$$(a)\quad F = \frac{1\times10^{-9}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)}\left[\frac{5\times10^{-9}[(1,-3,7)-(2,0,4)]}{[(1,-3,7)-(2,0,4)]^3} + \frac{(-2\times10^{-9})[(1,-3,7)-(-3,0,5)]}{[(1,-3,7)-(-3,0,5)]^3}\right]$$

$$= [\frac{45(-1,-3,3)}{19^{3/2}} - \frac{18(4,-3,2)}{29^{3/2}}] \text{ nN}$$

$$= \underline{\underline{-1.004\boldsymbol{a}_x - 1.284\boldsymbol{a}_y + 1.4\boldsymbol{a}_z \text{ nN}}}$$

$$(b)\quad E = \frac{F}{Q} = \underline{\underline{-1.004\boldsymbol{a}_x - 1.284\boldsymbol{a}_y + 1.4\boldsymbol{a}_z \text{ V/m}}}$$

P. E. 4.2

Let q be the charge on each sphere, i.e. q=Q/3. The free body diagram below helps us to establish the relationship between various forces.