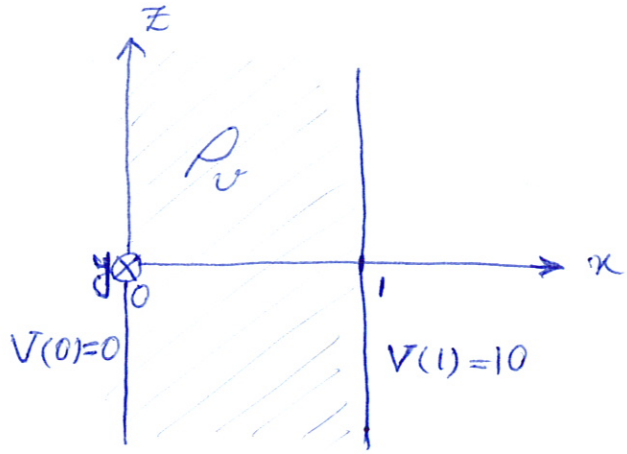
6.4 uniform charge over $0 < x < 1$; $\rho_v = -6\varepsilon$

$$\begin{cases} V(0) = 0 \\ V(1) = 10 \end{cases} \Rightarrow V(x) = ?$$



$V = V(x)$ ← Symmetry of Problem

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = -\frac{-6\varepsilon}{\varepsilon} = 6$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 6$$

$$V = V(x) \Rightarrow \frac{d^2 V}{dx^2} = 6$$

$$\Rightarrow \frac{dV}{dx} = 6x + C_1$$

$$\Rightarrow V = 3x^2 + C_1 x + C_2$$

$$\begin{cases} V(0) = 0 \Rightarrow V(0) = 3 \times 0 + C_1 \times 0 + C_2 = C_2 = 0 \\ V(1) = 10 \Rightarrow V(1) = 3 \times 1^2 + C_1 \times 1 + 0 = 3 + C_1 = 10 \Rightarrow C_1 = 7 \end{cases}$$

$$\Rightarrow V = V(x) = 3x^2 + 7x \ \text{V}$$

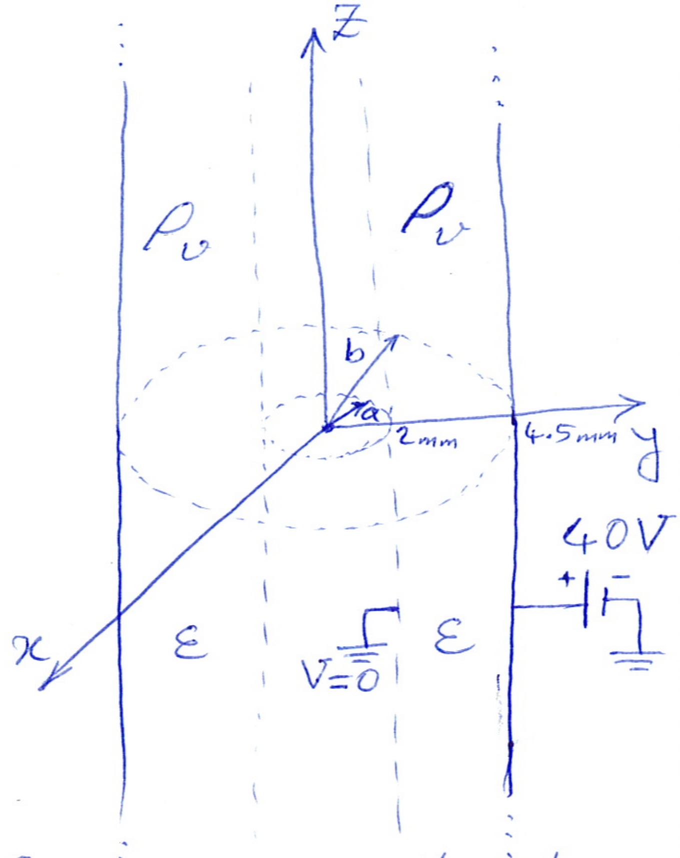
6.6

$$\rho_v = \frac{10\varepsilon_0}{\rho} \ C/m^3$$

$$\varepsilon = \varepsilon_0$$

$$a = 2mm$$
$$b = 4.5\,mm$$

$a < \rho < b \Rightarrow V = ?$



$$V = V(\rho)$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$V = V(\rho) \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \underbrace{\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}}_{=0} = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right)$$

$$= -\frac{\rho_v}{\varepsilon}$$

$$\Rightarrow \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = -\frac{\frac{10\varepsilon_0}{\rho}}{\varepsilon_0} = -\frac{10}{\rho}$$

$$\Rightarrow \frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = -10$$

$$\Rightarrow \rho\frac{dV}{d\rho} = -10\rho + C_1$$

$$\Rightarrow \frac{dV}{d\rho} = -10 + \frac{C_1}{\rho}$$

$$\Rightarrow V(\rho) = -10\rho + C_1 \ln \rho + C_2$$

$$\begin{cases} V(a) = -10a + C_1 \ln a + C_2 = 0 \\ V(b) = -10b + C_1 \ln b + C_2 = 40 \end{cases}$$

6.6

$$V(b) - V(a) = -10(b-a) + C_1(\underbrace{\ln b - \ln a}_{=\ln\left(\frac{b}{a}\right)}) = 40$$

$$\Rightarrow C_1 = \frac{40 + 10(b-a)}{\ln\left(\frac{b}{a}\right)}$$

$$V(a) = 0 \Rightarrow C_2 = 10a - C_1 \ln a = 10a - \frac{40+10(b-a)}{\ln\left(\frac{b}{a}\right)} \times \ln a$$

$$\begin{cases} a = 2\ \text{mm} \\ b = 4.5\ \text{mm} \end{cases}$$

$$\Rightarrow C_1 = \frac{40 + 10(4.5-2)\times 10^{-3}}{\ln\left(\frac{4.5\times 10^{-3}}{2\times 10^{-3}}\right)} = \frac{40 + 0.025}{\ln(2.25)} = 49.3569$$

$$C_2 = 10\times 2\times 10^{-3} - C_1 \times \ln(2\times 10^{-3}) = 306.7538$$

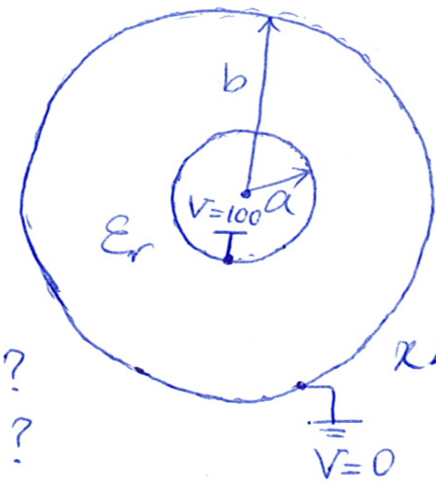
$$\Rightarrow V = V(\rho) = -10\rho + 49.3569 \ln \rho + 306.7538 \quad V$$

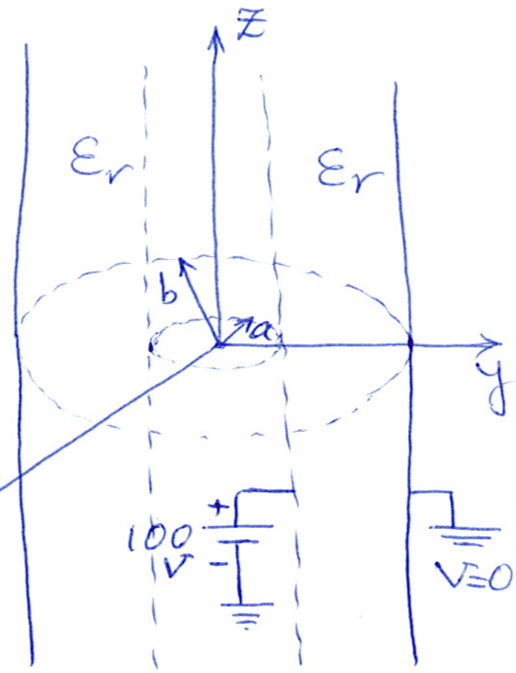
6.12

$$\begin{cases} a = 5 \text{ mm} \\ b = 15 \text{ mm} \end{cases}$$

$$\begin{cases} V(a) = 100 \text{ V} \\ V(b) = 0 \text{ V} \end{cases}$$

$$\varepsilon_r = 2.0$$





at $\rho = 10$ mm $\Rightarrow \begin{cases} V = ? \\ \underline{E} = ? \\ \underline{D} = ? \end{cases}$

at $\left.\begin{matrix} \rho = 5 \text{ mm} \\ \& \\ \rho = 15 \text{ mm} \end{matrix}\right\} \Rightarrow \rho_s = ?$ charge density on each plate

Assume the capacitor is tall enough that $V = V(\rho)$

$$\rho_v = 0 \Rightarrow \nabla^2 V = 0$$

$$V = V(\rho) \Rightarrow \nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

$$\Rightarrow \frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

$$\Rightarrow \rho \frac{dV}{d\rho} = C_1$$

$$\Rightarrow \frac{dV}{d\rho} = \frac{C_1}{\rho}$$

$$\Rightarrow V(\rho) = C_1 \ln \rho + C_2$$

6.12

$$\begin{cases} V(a) = C_1 \ln(a) + C_2 = 100 \\ V(b) = C_1 \ln(b) + C_2 = 0 \end{cases}$$

$$V(a) - V(b) = C_1 \ln\left(\frac{a}{b}\right) = 100 \quad \Rightarrow C_1 = \frac{100}{\ln\left(\frac{a}{b}\right)}$$

$$C_2 = -C_1 \ln(b) = -\frac{100}{\ln\left(\frac{a}{b}\right)} \times \ln(b) = \frac{-100 \ln(b)}{\ln\left(\frac{a}{b}\right)}$$

$$\Rightarrow V = V(\rho) = \frac{100}{\ln\left(\frac{a}{b}\right)} \ln\rho - \frac{100}{\ln\left(\frac{a}{b}\right)} \ln b = \frac{100 \ln\left(\frac{\rho}{b}\right)}{\ln\left(\frac{a}{b}\right)}$$

$$\Rightarrow V(\rho) = \frac{100 \ln\left(\frac{\rho}{b}\right)}{\ln\left(\frac{a}{b}\right)} = \frac{100 \ln(b/\rho)}{\ln(b/a)} \quad V$$

$$\underline{E} = -\nabla V = -\frac{\partial}{\partial \rho} V \underline{a}_\rho$$

$$= -\underline{a}_\rho \frac{\partial}{\partial \rho} \left(\frac{100 \ln\left(\frac{b}{\rho}\right)}{\ln\left(\frac{b}{a}\right)} \right)$$

$$= \frac{100}{\ln\left(\frac{b}{a}\right)} \underline{a}_\rho \underbrace{\frac{\partial}{\partial \rho} \Big[\underbrace{\ln\left(\frac{\rho}{b}\right)}_{= \ln(\rho) - \ln(b)} \Big]}_{= \frac{1}{\rho}}$$

$$= \frac{100 \underline{a}_\rho}{\rho \ln\left(\frac{b}{a}\right)}$$

$$\Rightarrow \underline{E} = \frac{100}{\rho \ln\left(\frac{b}{a}\right)} \underline{a}_\rho$$

$$\underline{D} = \varepsilon \underline{E} = \varepsilon_r \varepsilon_0 \underline{E} = 2 \varepsilon_0 \underline{E} = \frac{200 \varepsilon_0}{\rho \ln\left(\frac{b}{a}\right)} \underline{a}_\rho$$

6.12

at $\rho = 10$ mm $\Rightarrow$

$$\begin{cases} V(\rho=10\text{ mm}) = \dfrac{100 \ln\left(\frac{15}{10}\right)}{\ln\left(\frac{15}{5}\right)} = 36.9070 \text{ V} \\[2ex] \underline{E}(\rho=10\text{ mm}) = \dfrac{100}{10\times10^{-3}\times\ln\left(\frac{15}{5}\right)}\underline{a}_\rho = 9102.39\,\underline{a}_\rho \text{ V/m} \\[2ex] \underline{D}(\rho=10\text{ mm}) = \dfrac{200\times\frac{1}{36\pi}\times10^{-9}}{10\times10^{-3}\times\ln\left(\frac{15}{5}\right)}\underline{a}_\rho = 160.9656\,\underline{a}_\rho \text{ nC/m}^2 \end{cases}$$

$$\rho_s = D_n \quad \Leftarrow \text{ Conductor Boundary Conditions}$$

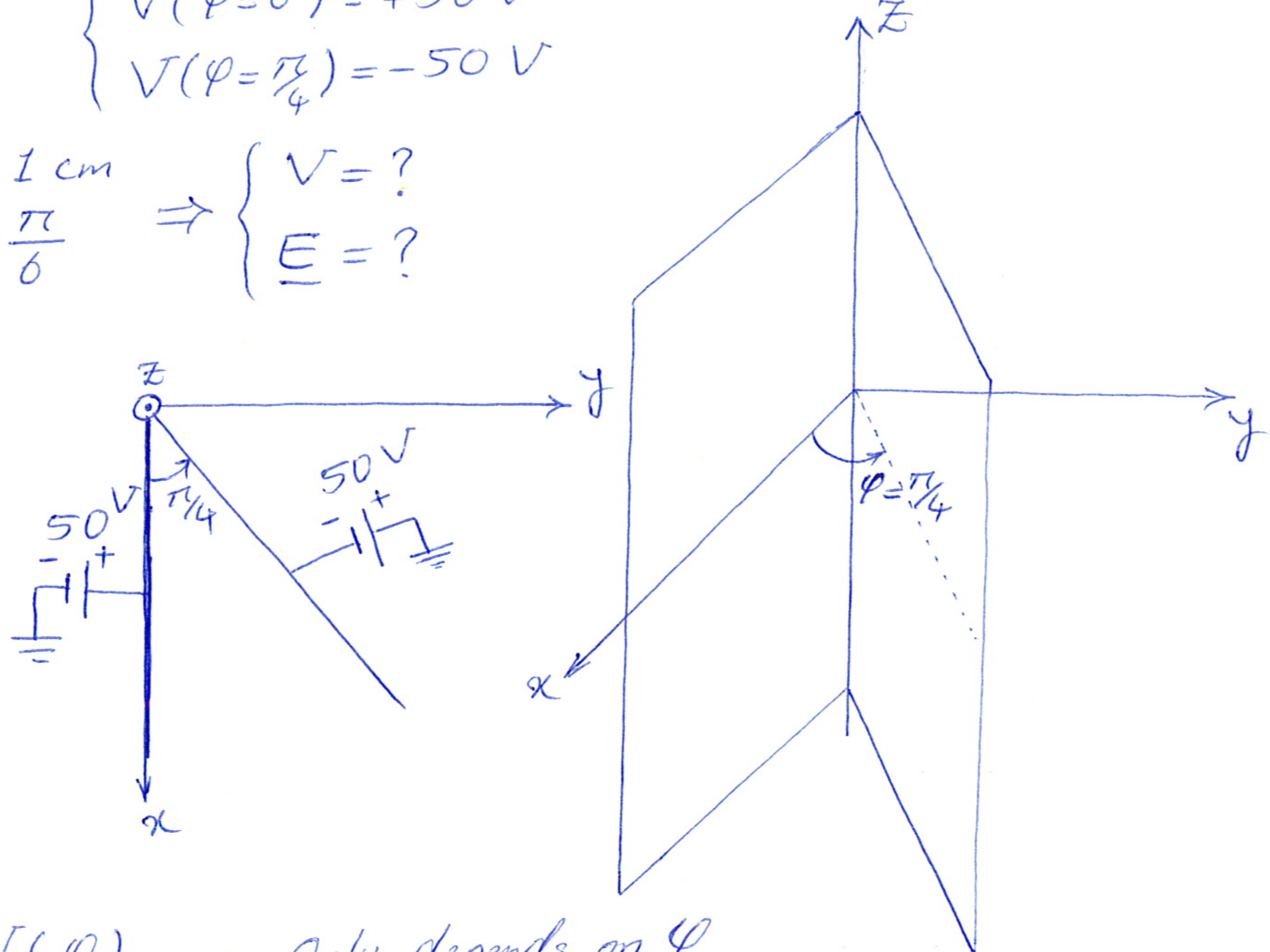
$$= \underline{D}\cdot\underline{a}_n$$

$$\begin{cases} \text{at } \rho = 5 \text{ mm} \Rightarrow \underline{a}_n = \underline{a}_\rho \Rightarrow \rho_s = \underline{D}\cdot\underline{a}_\rho\big|_{\rho=5\text{mm}} \\ \text{at } \rho = 15 \text{ mm} \Rightarrow \underline{a}_n = -\underline{a}_\rho \Rightarrow \rho_s = -\underline{D}\cdot\underline{a}_\rho\big|_{\rho=15\text{mm}} \end{cases}$$

$$\begin{cases} \rho_s(\rho=5\text{ mm}) = \dfrac{200\times\frac{1}{36\pi}\times10^{-9}}{5\times10^{-3}\times\ln\left(\frac{15}{5}\right)} = 321.93 \text{ nC/m}^2 \\[3ex] \rho_s(\rho=15\text{ mm}) = \dfrac{-200\times\frac{1}{36\pi}\times10^{-9}}{15\times10^{-3}\times\ln\left(\frac{15}{5}\right)} = -107.31 \text{ nC/m}^2 \end{cases}$$

6.14 $\begin{cases} V(\varphi=0) = +50\text{ V} \\ V(\varphi=\frac{\pi}{4}) = -50\text{ V} \end{cases}$

at $\begin{cases} \rho = 1\text{ cm} \\ \varphi = \frac{\pi}{6} \end{cases} \Rightarrow \begin{cases} V = ? \\ \underline{E} = ? \end{cases}$



$V = V(\varphi)$ ← Only depends on φ

$$\Rightarrow \nabla^2 V = \frac{1}{\rho^2}\frac{d^2V}{d\varphi^2} = 0$$

$$\Rightarrow \frac{d^2V}{d\varphi^2} = 0$$

$$\Rightarrow \frac{dV}{d\varphi} = C_1$$

$$\Rightarrow V(\varphi) = C_1\varphi + C_2$$

$$\begin{cases} V(\varphi=0) = C_1 \times 0 + C_2 = 50 & \Rightarrow C_2 = 50 \\ V(\varphi=\frac{\pi}{4}) = C_1 \times \frac{\pi}{4} + C_2 = -50 & \Rightarrow C_1 = \frac{-400}{\pi} \end{cases}$$

$$\Rightarrow V(\varphi) = \frac{-400}{\pi}\varphi + 50 \quad \text{V}$$

6.14

$$\underline{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\varphi}\underline{a}_\varphi$$

$$= -\frac{1}{\rho}\frac{d}{d\varphi}\left[\frac{-400}{\pi}\varphi + 50\right]\underline{a}_\varphi$$

$$= \frac{400}{\pi\rho}\underline{a}_\varphi$$

$$\Rightarrow \underline{E} = \frac{400}{\pi\rho}\underline{a}_\varphi$$

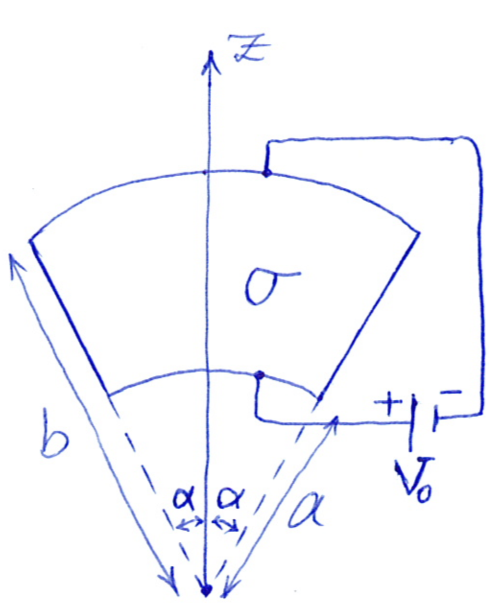
$$\text{at} \begin{cases} \rho = 1\,cm = 10^{-2}\ m \\ \varphi = \frac{\pi}{6} \end{cases}$$

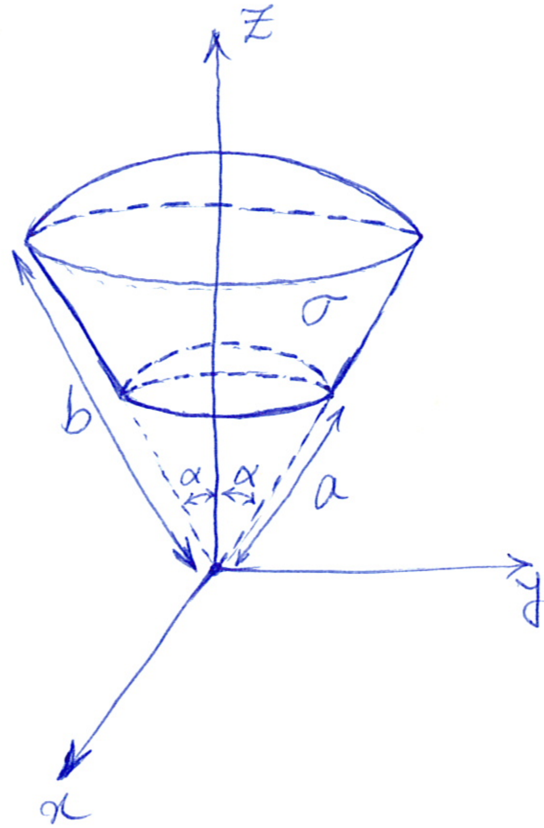
$$V\left(\varphi = \frac{\pi}{6}\right) = \frac{-400}{\cancel{\pi}} \times \frac{\cancel{\pi}}{6} + 50 = -16.6667\ V$$

$$\underline{E}\left(\rho = 10^{-2}, \varphi = \frac{\pi}{6}\right) = \frac{400}{\pi \times 10^{-2}}\underline{a}_\varphi = 12.7324\ \underline{a}_\varphi\ KV/m$$

6.24

$$\begin{cases} a < r < b \\ 0 < \theta < \alpha \\ 0 < \varphi < 2\pi \end{cases}$$





$$\begin{cases} V(r=a) = V_0 \\ V(r=b) = 0 \end{cases}$$

$$R \overset{?}{=} \frac{1}{2\pi\sigma(1-\cos\alpha)}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Assume that the spherical shell is thin enough that $V = V(r)$ only.

$$\nabla^2 V = 0$$

$$V = V(r) \Rightarrow \nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

$$\Rightarrow r^2 \frac{dV}{dr} = C_1$$

$$\Rightarrow \frac{dV}{dr} = \frac{C_1}{r^2}$$

$$\Rightarrow V(r) = \frac{-C_1}{r} + C_2$$

6.24

$$\begin{cases} V(r=a) = \dfrac{-C_1}{a} + C_2 = V_0 \\ V(r=b) = \dfrac{-C_1}{b} + C_2 = 0 \quad \Rightarrow C_2 = \dfrac{C_1}{b} \end{cases}$$

$$\Rightarrow V(r=a) - V(r=b) = C_1\left(\frac{1}{b} - \frac{1}{a}\right) = V_0 \Rightarrow C_1 = \frac{V_0}{\frac{1}{b} - \frac{1}{a}}$$

$$\Rightarrow C_2 = \frac{V_0}{b\left[\frac{1}{b} - \frac{1}{a}\right]}$$

$$\Rightarrow V(r) = \frac{-V_0}{r\left[\frac{1}{b} - \frac{1}{a}\right]} + \frac{V_0}{b\left[\frac{1}{b} - \frac{1}{a}\right]} = \frac{-V_0}{\left[\frac{1}{b} - \frac{1}{a}\right]}\left[\frac{1}{r} - \frac{1}{b}\right]$$

$$\underline{E} = -\nabla V = -\frac{dV}{dr}\underline{a}_r$$

$$= \frac{+V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)}\underline{a}_r \underbrace{\frac{d}{dr}\left(\frac{1}{r} - \frac{1}{b}\right)}_{=\frac{-1}{r^2}}$$

$$= \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2}\underline{a}_r$$

$$\underline{J} = \sigma \underline{E} = \frac{\sigma V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2}\underline{a}_r$$

$$I = \iint_S \underline{J} \cdot d\underline{s} \longleftarrow \begin{cases} S: \begin{cases} 0 < \theta < \alpha \\ 0 < \varphi < 2\pi \end{cases} \\ d\underline{s} = r^2 \sin\theta \, d\theta \, d\varphi \, \underline{a}_r \end{cases}$$

$$= \int_{\theta=0}^{\alpha} \int_{\varphi=0}^{2\pi} \left(\frac{\sigma V_0}{\left[\frac{1}{a} - \frac{1}{b}\right]r^2}\underline{a}_r\right) \cdot \left(r^2 \sin\theta \, d\theta \, d\varphi \, \underline{a}_r\right)$$

6.24

$$I = \int_{\theta=0}^{\alpha} \int_{\varphi=0}^{2\pi} \frac{\sigma V_0}{\left[\frac{1}{a}-\frac{1}{b}\right] r^2} \times r^2 \sin\theta \, d\theta \, d\varphi$$

$$= \frac{\sigma V_0}{\left[\frac{1}{a}-\frac{1}{b}\right]} \int_{\theta=0}^{\alpha} \sin\theta \, d\theta \int_{\varphi=0}^{2\pi} d\varphi$$

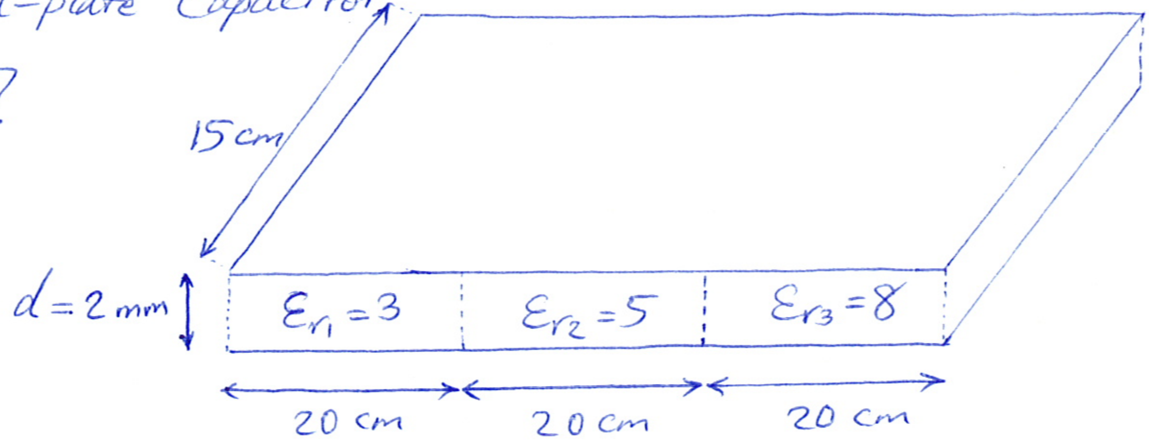
$$= \frac{\sigma V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)} \times (-\cos\theta)\Big|_{\theta=0}^{\alpha} \times \varphi\Big|_{\varphi=0}^{2\pi}$$

$$= \frac{\sigma V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)} (1-\cos\alpha) \times 2\pi$$

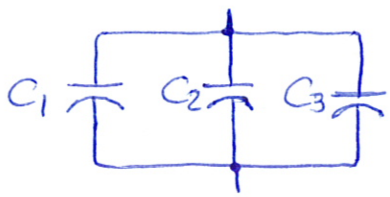
$$R = \frac{V_0}{I} = \frac{\left(\frac{1}{a}-\frac{1}{b}\right)}{2\pi\sigma(1-\cos\alpha)}$$

6.28 Parallel-plate Capacitor

$C = ?$



This problem can be considered as three parallel capacitor:



$$C_{tot} = C_1 + C_2 + C_3$$

The capacitance of parallel-plate capacitor is:

$$C = \frac{\varepsilon S}{d}$$

So, we have:

$$C_1 = \frac{\varepsilon_1 S_1}{d}, \quad C_2 = \frac{\varepsilon_2 S_2}{d}, \quad C_3 = \frac{\varepsilon_3 S_3}{d}$$

And total capacitance is equal to:

$$C_{tot} = \sum_{i=1}^{3} \frac{\varepsilon_i S_i}{d} = \sum_{i=1}^{3} \frac{\varepsilon_{ri} \varepsilon_0 S_i}{d}$$

Here:

$$S_1 = S_2 = S_3 = 0.20 \times 0.15 = 0.03 = S$$

$$\Rightarrow C_{tot} = \frac{\varepsilon_0 S}{d} (\varepsilon_{r1} + \varepsilon_{r2} + \varepsilon_{r3})$$

$$= \frac{\frac{1}{36\pi} \times 10^{-9} \times 0.03}{2 \times 10^{-3}} (\underbrace{3 + 5 + 8}_{=16}) = 2.122\ nF$$

$$\Rightarrow C_{tot} = 2.122\ nF$$

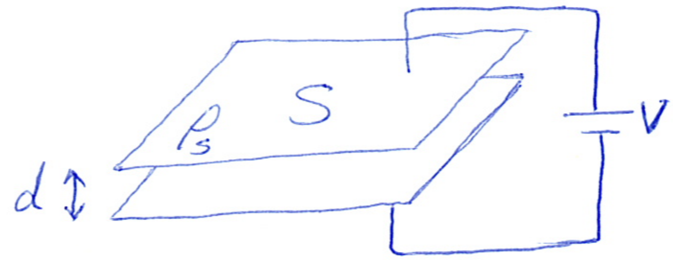
6.33 parallel-plate capacitor

$S = 200\ cm^2 = 200 \times 10^{-4}\ m^2$

$d = 3mm = 3 \times 10^{-3}\ m$

$\varepsilon = \varepsilon_0 \Leftarrow$ air

$\rho_s = 1\ \mu C/m^2$



a) $C = ?$

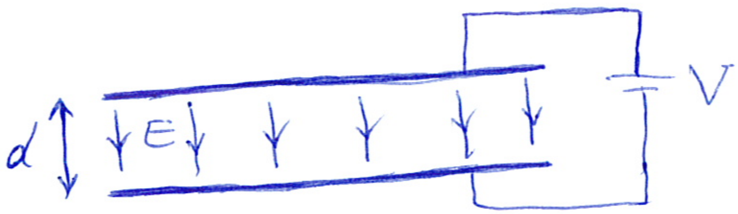
$$C = \frac{\varepsilon S}{d} = \frac{\varepsilon_0 \times S}{d}$$

$$= \frac{\frac{1}{36\pi} \times 10^{-9} \times 200 \times 10^{-4}}{3 \times 10^{-3}} = 58.9463 \times 10^{-12}\ F$$

$$= 58.9463\ pF$$

$\Rightarrow C = 58.9463\ pF$

b) $V = ?$



$$E = \frac{V}{d} \Rightarrow V = Ed$$

$$E = \frac{D}{\varepsilon} = \frac{D}{\varepsilon_0} \Rightarrow V = \frac{D}{\varepsilon_0} d$$

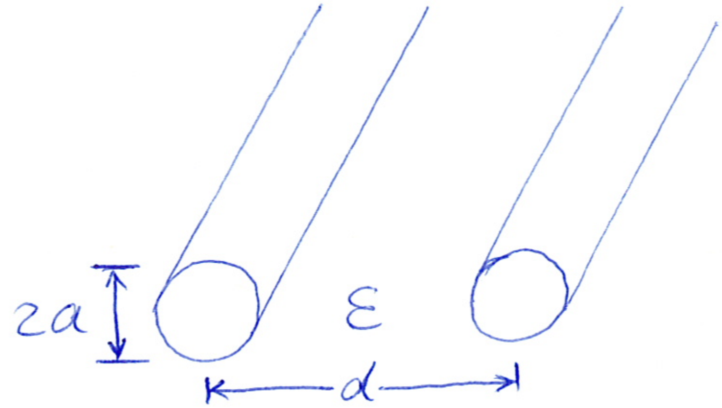
$D = D_n = \rho_s \Leftarrow$ Boundary Conditions

$$\Rightarrow V = \frac{\rho_s}{\varepsilon_0} d = \frac{10^{-6} \times 3 \times 10^{-3}}{\frac{1}{36\pi} \times 10^{-9}} = 108\pi\ V$$

$\Rightarrow V = 339.292\ V$

6.43

$$C = \frac{\pi \varepsilon}{\cosh^{-1}\left(\frac{d}{2a}\right)}$$



$$G = \frac{1}{R} = ?$$

$$RC = \frac{\varepsilon}{\sigma} \Rightarrow \frac{C}{G} = \frac{\varepsilon}{\sigma}$$

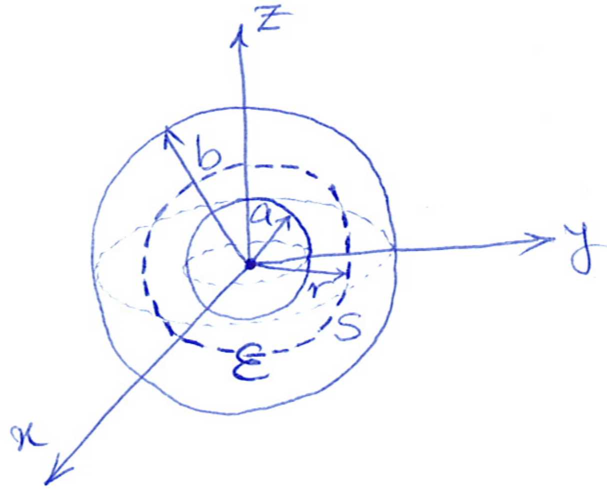
$$\Rightarrow G = \frac{\sigma}{\varepsilon} C$$

$$\Rightarrow G = \frac{\pi \sigma}{\cosh^{-1}\left(\frac{d}{2a}\right)}$$

6.46 Spherical Capacitor

$$\varepsilon = \frac{\varepsilon_0 k}{r^2}$$

$$C \stackrel{?}{=} \frac{4\pi\varepsilon_0 k}{b-a}$$



Let's assume charge Q is on inner sphere.

$$Q_{enc} = Q = \oint\!\!\oint_S \underline{D} \cdot d\underline{s} = D_r \oint\!\!\oint_S ds = D_r 4\pi r^2$$

$$\Rightarrow \underline{D} = \frac{Q}{4\pi r^2} \underline{a_r}$$

$$\Rightarrow \underline{E} = \frac{\underline{D}}{\varepsilon} = \frac{\frac{Q}{4\pi r^2}\underline{a_r}}{\frac{\varepsilon_0 k}{r^2}} = \frac{Q}{4\pi\varepsilon_0 k}\underline{a_r}$$

$$V = \int_a^b \underline{E} \cdot d\underline{l} = \int_{r=a}^{b} \left(\frac{Q}{4\pi\varepsilon_0 k}\underline{a_r}\right) \cdot (\underline{a_r} dr) \qquad d\underline{l} = \underline{a_r} dr$$

$$= \frac{Q}{4\pi\varepsilon_0 k} \int_{r=a}^{b} dr$$

$$= \frac{Q(b-a)}{4\pi\varepsilon_0 k}$$

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon_0 k}{b-a}$$