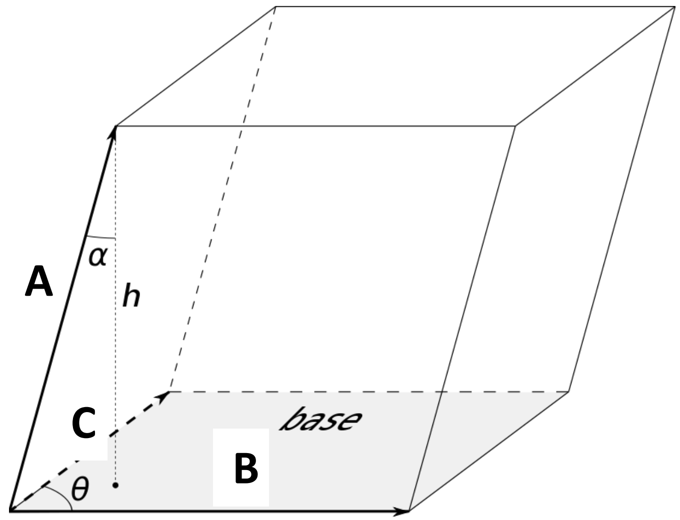
Lecture 2: Coordinate Systems and Transformations

Scalar triple product, vector triple product, Cartesian coordinates, cylindrical coordinates, transformations between Cartesian and Cylindrical, Chapter 1: pages 15-25, Chapter 2: pages 29-33

Triple Scalar Product



Wikipedia

This product, as the name implies, gives a scalar product of 3 vectors

$$\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C})=\mathbf{B}\bullet(\mathbf{C}\times\mathbf{A})=\mathbf{C}\bullet(\mathbf{A}\times\mathbf{B})$$ (notice the cyclic expression)

This product gives the volume of the parallelogram whose edges are the three vectors

Triple Scalar Product (Cont'd)

This scalar product can be shown to be given by the determinant

$$\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= A_x(B_yC_z - B_zC_y) - A_y(B_xC_z - B_zC_x) + A_z(B_xC_y - B_yC_x)$$

Vector Triple Product

As the name implies, the result of this product is a vector using 3 other vectors

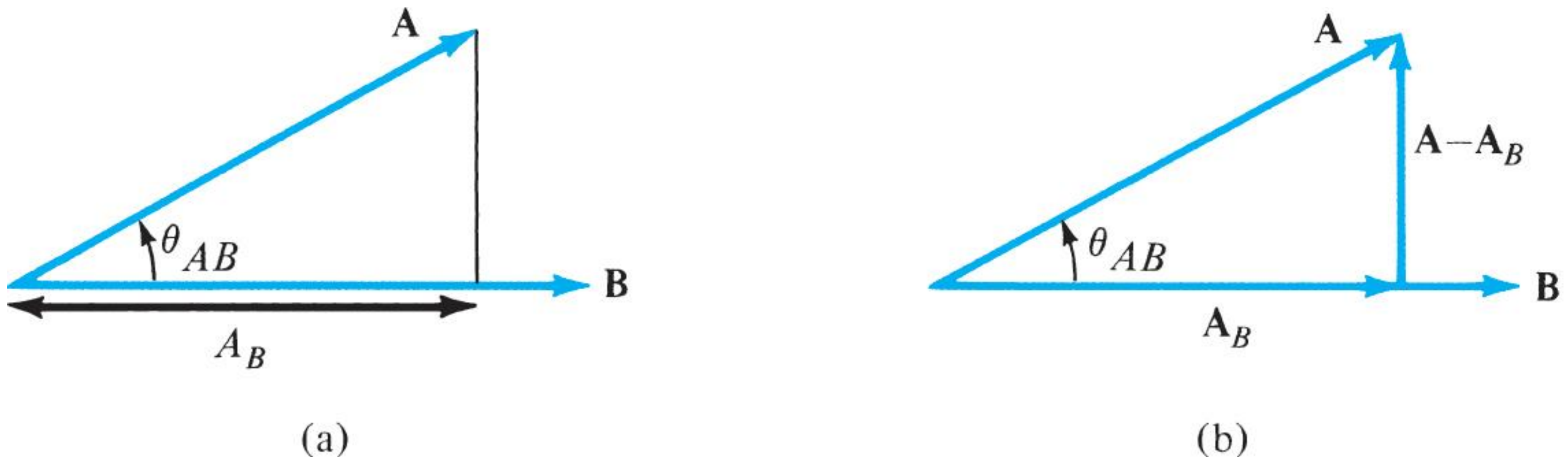
$$\mathrm{VTP} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \bullet \mathbf{C}) - \mathbf{C}(\mathbf{A} \bullet \mathbf{B})$$

Related identities

$$\mathbf{A}(\mathbf{B} \bullet \mathbf{C}) \neq (\mathbf{A} \bullet \mathbf{B})\mathbf{C}$$

$$\mathbf{C}(\mathbf{A} \bullet \mathbf{B}) = (\mathbf{A} \bullet \mathbf{B})\mathbf{C}$$

Components of a vector

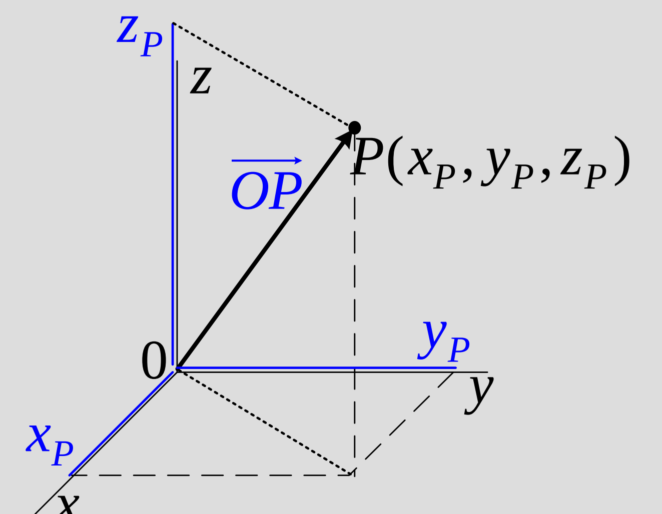


The projection of a vector **A** in the direction of a vector **B** is given by $A_B = \mathbf{A}.\mathbf{a}_B$, where $\mathbf{a}_B$ is the unit vector in the direction of **B**

The vector projection of **A** in the direction of B is thus $\mathbf{A}_B = (\mathbf{A}.\mathbf{a}_B)\,\mathbf{a}_B$

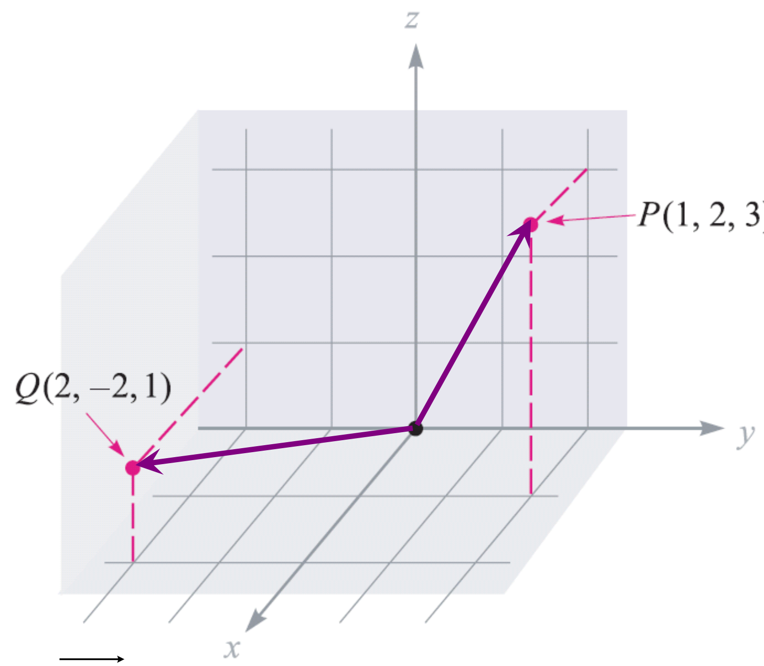
The vector component of **A** normal to **B** is $\mathbf{D} = \mathbf{A} - \mathbf{A}_B$

Cartesian (Rectangular) Coordinate System



position coordinates: (x_P, y_P, z_P)

$$\mathbf{R}_P = x_P\mathbf{a}_x + y_P\mathbf{a}_y + z_P\mathbf{a}_z$$

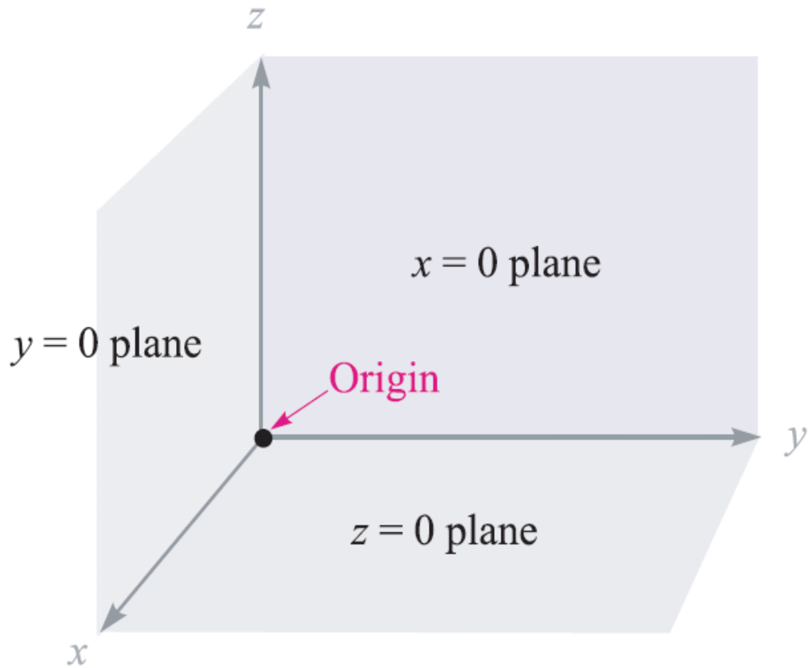


$$\overrightarrow{OP} \equiv \mathbf{R}_P = 1\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$$
$$\overrightarrow{OQ} \equiv \mathbf{R}_Q = 2\mathbf{a}_x - 2\mathbf{a}_y + 1\mathbf{a}_z$$

An origin and three orthogonal axis are first determined

Any point is determined by the intersection of 3 orthogonal planes

Cartesian Coordinates (Cont'd)



principal planes satisfy: $x=const.$, $y=const.$, $z=const.$

principal lines are intersections of two principal planes:

$$x = const.,\ y = const.\ (z\ \text{varies})$$
$$y = const.,\ z = const.\ (x\ \text{varies})$$
$$z = const.,\ x = const.\ (y\ \text{varies})$$

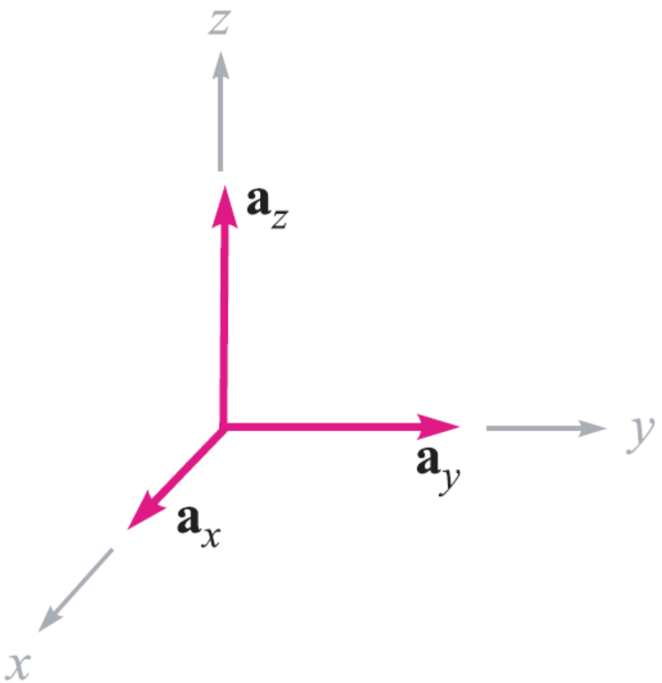
Where are these lines?

line $x = 0,\ y = 0$

line $y = 0,\ z = 0$

line $z = 0,\ x = 0$

Cartesian Coordinates (Cont'd)



$-\infty < x < \infty$

$-\infty < y < \infty$

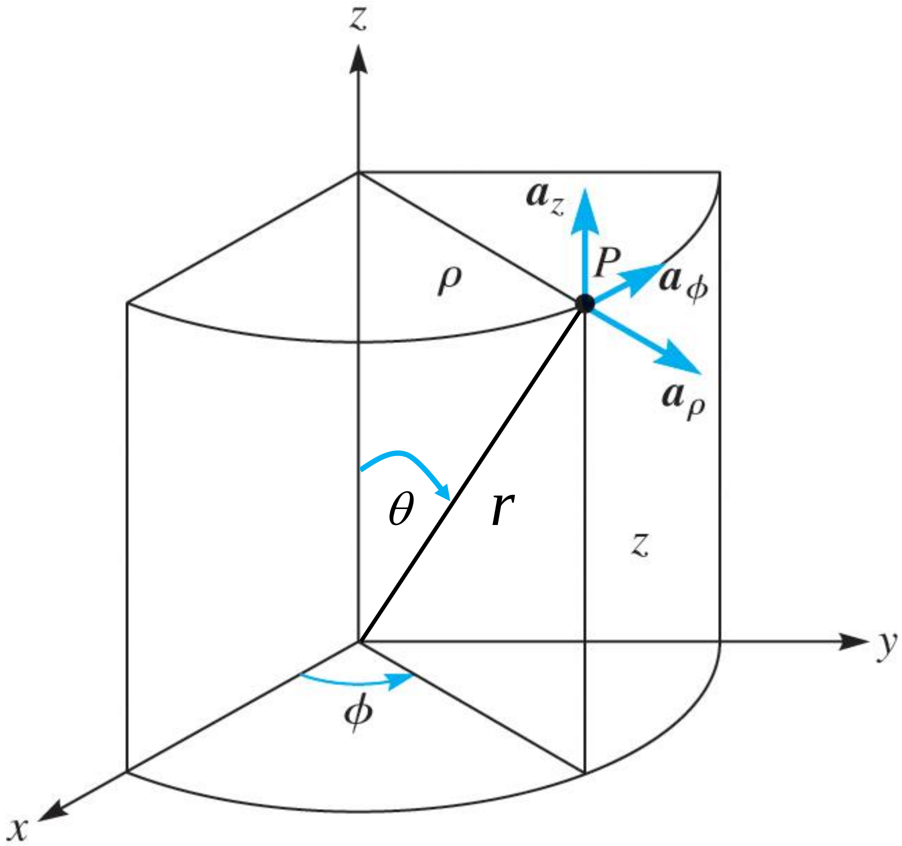
$-\infty < z < \infty$

Directions of coordinate axes $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ do not change from one point to another

The position vector of any point $(x,y,z) = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

Any vector with components (A_x, A_y, A_z) is written as $\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$

Cylindrical Coordinates



$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\rho = r \sin\theta$$

$$x = \rho\cos\varphi$$

$$y = \rho\sin\varphi$$

position coordinates (ρ, φ, z)

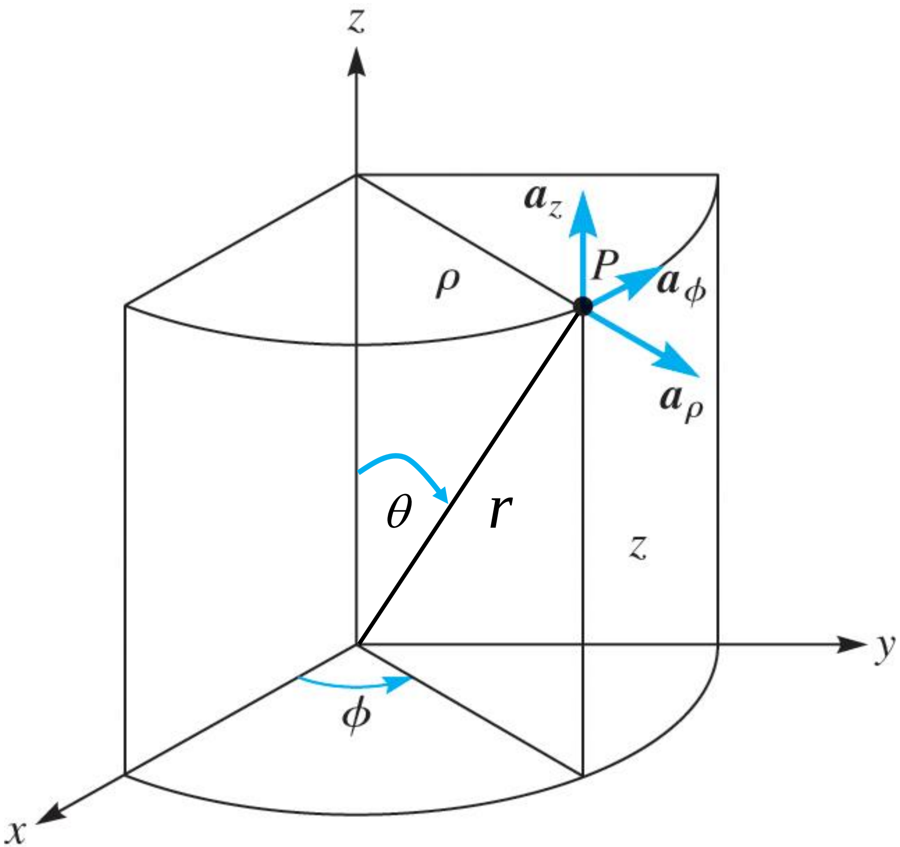
position vector $\mathbf{R} = \rho\mathbf{a}_\rho + z\mathbf{a}_z$

No φ component (why?)

Notice that $\mathbf{a}_\rho$ changes from one point to another as a function of φ!

Notice also that $\mathbf{a}_\varphi$ is normal to the plane containing $\mathbf{R}$ and $\mathbf{a}_z$

Cylindrical Coordinates (Cont'd)



$0 \le \rho,\ 0 \le \varphi \le 2\pi,\ -\infty < z < \infty$

Any vector (not a position vector) with components (A_ρ, A_φ, A_z) can be written as $\mathbf{A} = A_\rho\,\mathbf{a}_\rho + A_\varphi\,\mathbf{a}_\varphi + A_z\,\mathbf{a}_z$

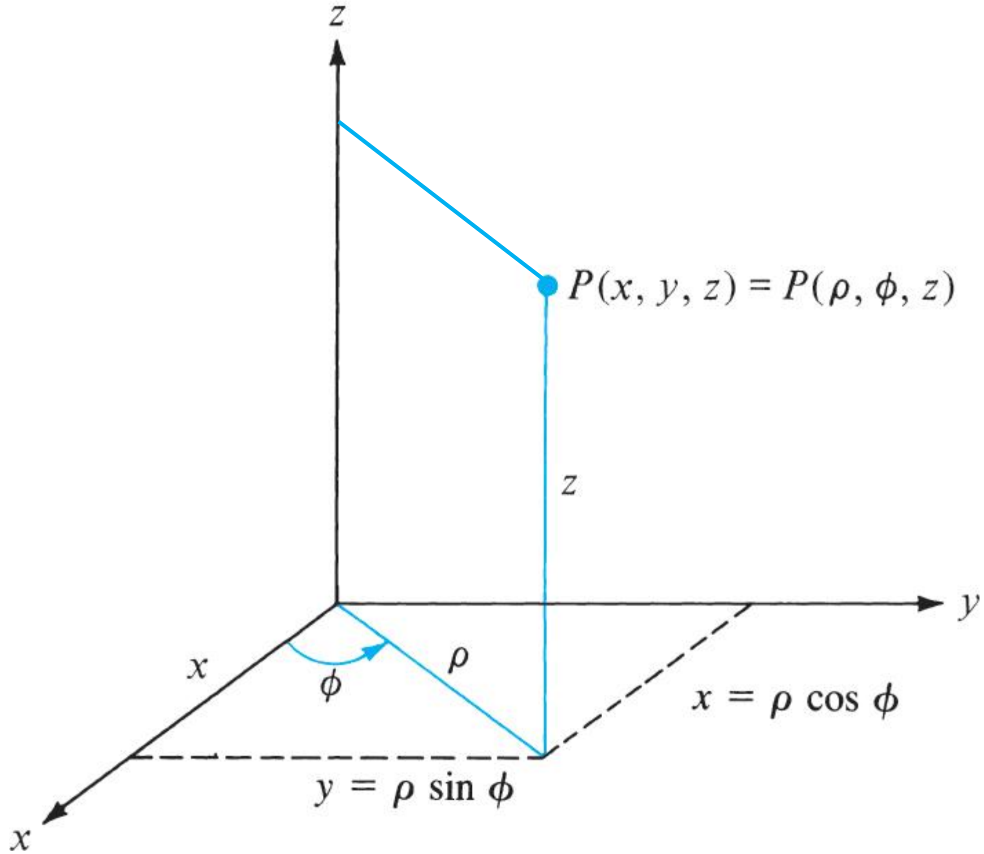
Because the cylindrical coordinates are mutually orthogonal, we have

$$|\mathbf{A}| = \sqrt{A_\rho^2 + A_\varphi^2 + A_z^2}$$

$$\mathbf{a}_\rho \cdot \mathbf{a}_\varphi = 0,\ \ \mathbf{a}_\rho \cdot \mathbf{a}_z = 0,\ \ \mathbf{a}_\varphi \cdot \mathbf{a}_z = 0,$$

$$\mathbf{a}_\rho \cdot \mathbf{a}_\rho = 1,\ \ \mathbf{a}_\varphi \cdot \mathbf{a}_\varphi = 1,\ \ \mathbf{a}_z \cdot \mathbf{a}_z = 1$$

Coordinate Transformations



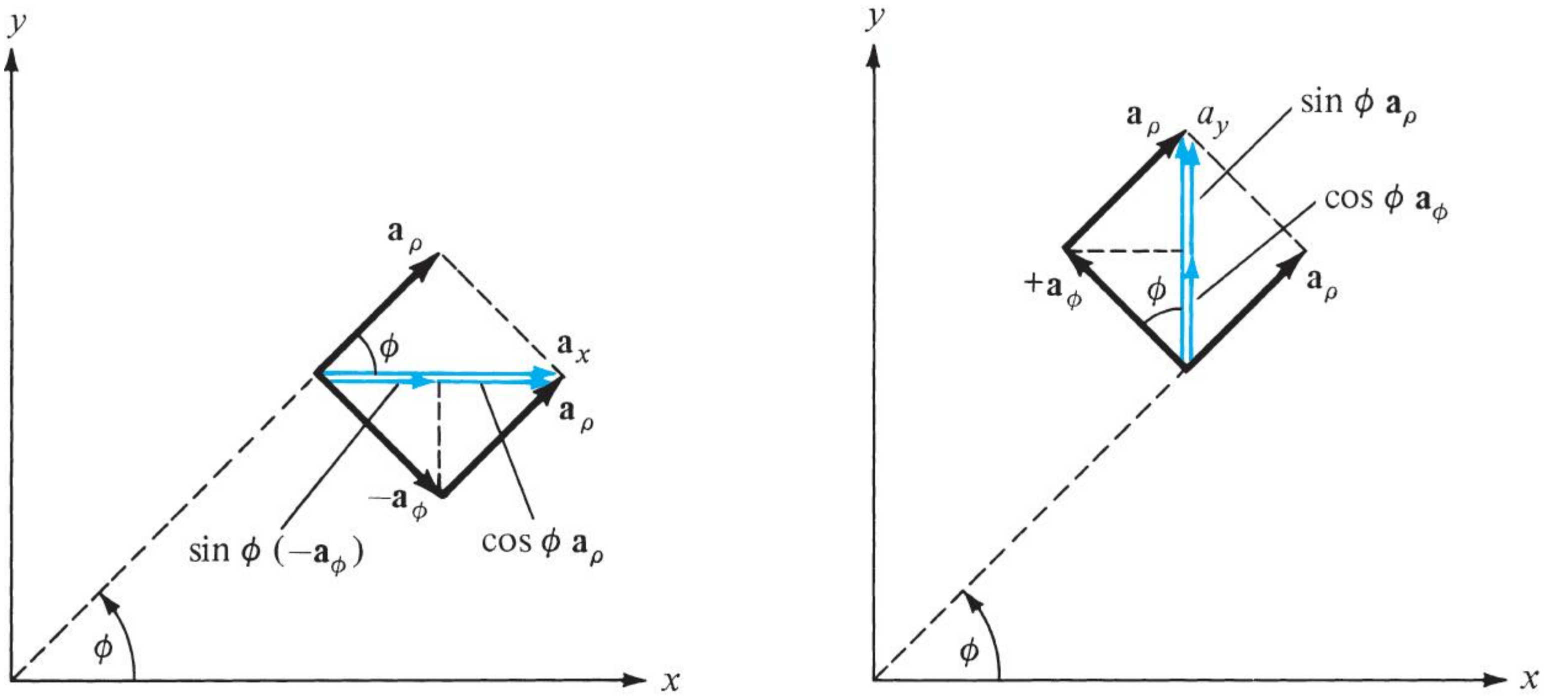
Cylindrical to Cartesian

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases}$$

Cartesian to Cylindrical

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \arctan\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

Vector Transformations



(a) (b)

$$\mathbf{a}_x = \cos\varphi\,\mathbf{a}_\rho - \sin\varphi\,\mathbf{a}_\varphi \qquad \mathbf{a}_y = \sin\varphi\,\mathbf{a}_\rho + \cos\varphi\,\mathbf{a}_\varphi$$

Given $\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$, what are ($A\rho$, $A\varphi$, Az)?

Vector Transformations (Cont'd)

$\mathbf{A}= A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$

$\mathbf{A}=A_x(\cos\varphi\,\mathbf{a}_\rho - \sin\varphi\,\mathbf{a}_\varphi) + A_y(\sin\varphi\,\mathbf{a}_\rho + \cos\varphi\,\mathbf{a}_\varphi) + A_z\mathbf{a}_z$

$$\mathbf{A}=(\underbrace{A_x\cos\varphi + A_y\sin\varphi}_{A_\rho})\mathbf{a}_\rho + (\underbrace{-A_x\sin\varphi + A_y\cos\varphi}_{A_\varphi})\mathbf{a}_\varphi + A_z\mathbf{a}_z$$

In matrix form, we have

$$\begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix}$$

Remember what φ means!