Lecture 28:Time-Varying Fields

Vector Potentials, Time-harmonic fields, Chapter 9, 417-429

The Statics Case

$$V = \iiint_V \frac{\rho_v \, dv}{4\pi\varepsilon R} \Rightarrow \nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$\mathbf{E} = -\nabla V$$

$$\mathbf{A} = \iiint_V \frac{\mu \mathbf{J} \, dv}{4\pi\varepsilon R}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

How do things change when charges and currents are varying with time?

Dynamic Electric Field

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial(\nabla \times \mathbf{A})}{\partial t} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = \mathbf{0} \;\Rightarrow\; \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V$$

Electric field is generated by both charges and time-varying currents!

Potentials

$$\nabla.\mathbf{D} = \rho_v \;\Rightarrow\; \nabla.\mathbf{E} = \frac{\rho_v}{\varepsilon} = -\nabla^2 V - \frac{\partial(\nabla.\mathbf{A})}{\partial t}$$

$$\nabla^2 V + \frac{\partial(\nabla.\mathbf{A})}{\partial t} = -\frac{\rho_v}{\varepsilon}$$

we need to eliminate the magnetic vector potential **A** term to obtain a differential equation in *V* alone. We start from Maxwell's second curl equation.

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \Rightarrow \nabla\times\mathbf{B} = \mu\mathbf{J} + \mu\frac{\partial\mathbf{D}}{\partial t} = \mu\mathbf{J} + \mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\times\nabla\times\mathbf{A} = \mu\mathbf{J} + \mu\varepsilon\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{A}}{\partial t} - \nabla V\right)$$

Potentials (Cont'd)

using the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla.\mathbf{A}) - \nabla^2 \mathbf{A}$$

we have

$$\nabla(\nabla.\mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} + \mu\varepsilon \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla V\right) = \mu \mathbf{J} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu\varepsilon \nabla \frac{\partial V}{\partial t}$$

choosing $\nabla.\mathbf{A} = -\mu\varepsilon \dfrac{\partial V}{\partial t}$ (Lorentz condition),
we have

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} + \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \;\Rightarrow\; \nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

Potentials' Summary

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

$$u = 1/\sqrt{\mu\varepsilon}$$

$$V(t) = \iiint_V \frac{\rho_v(t')\, dv}{4\pi\varepsilon R}$$

$$\mathbf{A}(t) = \iiint_V \frac{\mu \mathbf{J}(t')\, dv}{4\pi\varepsilon R}$$

$$t' = t - \frac{R}{u}$$

The Time-Harmonic case

if all sources (charges and currents) have a sinusoidal waveform with angular frequency ω then all vector components are also sinusoidal with the same frequency but with different amplitudes and phases

it follows that these field components are represented by their phasors and no need to include the time domain dependence

$$f(x,t) = 3.0\cos(\omega t-\beta x)=3.0\,\mathrm{Re}(\exp(j(\omega t-\beta x)))$$

$$f(x,t) = \mathrm{Re}(3.0\exp(-j\beta x)\exp(j\omega t))$$

$$\tilde{f} = 3.0\ \exp(-j\beta\, x), \quad f(x,t) = \mathrm{Re}(\tilde{f}\ \exp(j\omega\, t))$$

Time-Harmonic Case (Cont'd)

similarly, for all field quantities we may write

$$\mathbf{E}(x,y,z,t)=\mathrm{Re}(\tilde{\mathbf{E}}(x,y,z)\exp(j\omega t))$$

$$\mathbf{H}(x,y,z,t)=\mathrm{Re}(\tilde{\mathbf{H}}(x,y,z)\exp(j\omega t))$$

$$\mathbf{D}(x,y,z,t)=\mathrm{Re}(\tilde{\mathbf{D}}(x,y,z)\exp(j\omega t))$$

$$\mathbf{B}(x,y,z,t)=\mathrm{Re}(\tilde{\mathbf{B}}(x,y,z)\exp(j\omega t))$$

$$\mathbf{J}(x,y,z,t)=\mathrm{Re}(\tilde{\mathbf{J}}(x,y,z)\exp(j\omega t))$$

$$q(x,y,z,t)=\mathrm{Re}(\tilde{q}(x,y,z)\exp(j\omega t))$$

Integral Time-Harmonic Maxwell's Equations

Maxwell's equations for the time-harmonic case are obtained by replacing each time vector by its corresponding phasor vector and replacing $\partial/\partial t$ by $j\omega$

Maxwell's equations in the integral form are given by

$$\oiint_S \tilde{\mathbf{D}}.\mathbf{dS} = \iiint_V \tilde{q}_v \; dV = \tilde{Q}_v$$

$$\oiint_S \tilde{\mathbf{B}}.d\mathbf{S} = 0$$

$$\oint_C \tilde{\mathbf{E}}.d\mathbf{l} = -j\omega \iint_S \tilde{\mathbf{B}}.d\mathbf{S}$$

$$\oint_C \tilde{\mathbf{H}}.d\mathbf{l} = \iint_S \tilde{\mathbf{J}}.d\mathbf{S} + j\omega \iint_S \tilde{\mathbf{D}}.d\mathbf{S}$$

Differential Time-Harmonic Maxwell's Equations

$$\nabla.\tilde{\mathbf{D}} = \tilde{q}_v$$

$$\nabla.\tilde{\mathbf{B}} = 0$$

$$(\nabla \times \tilde{\mathbf{E}}) = -j\omega\tilde{\mathbf{B}}$$

$$(\nabla \times \tilde{\mathbf{H}}) = \tilde{\mathbf{J}} \ + \ j\omega\tilde{\mathbf{D}}$$

same boundary conditions apply!