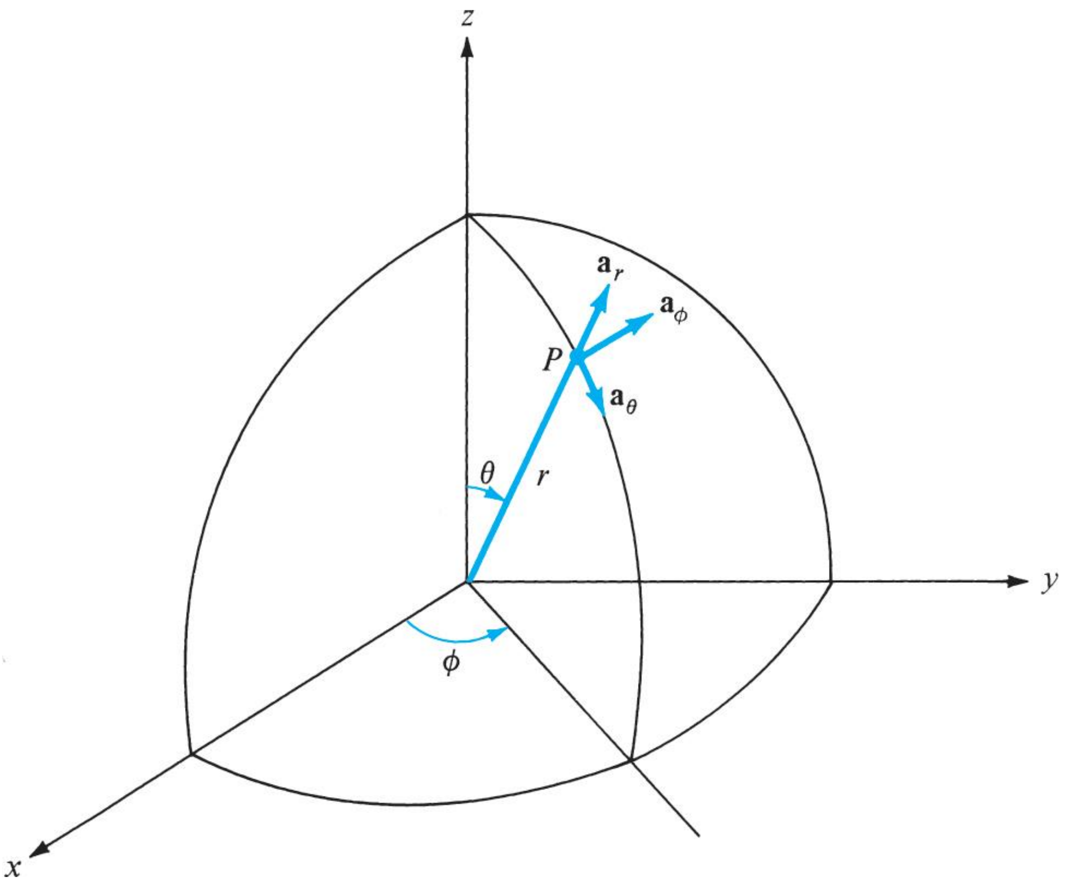
Lecture 3: Coordinate Systems and Transformations

Spherical coordinates, Constant value surfaces.

Chapter 2: pages 33-49

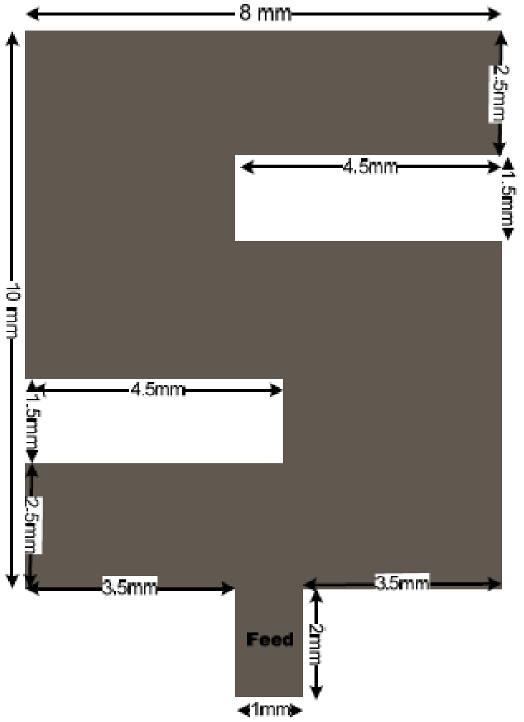
Spherical Coordinate System

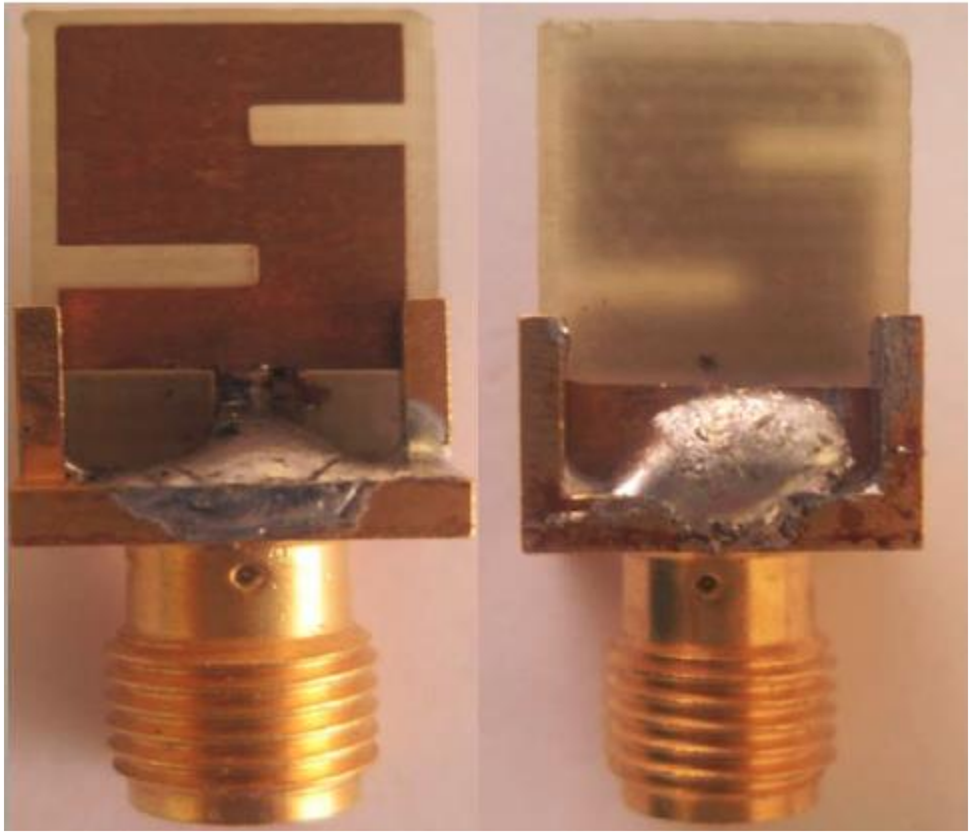


Every vector is given by 3 components A_r, A_θ, and A_φ

Position vector has only an *r* component (why?)

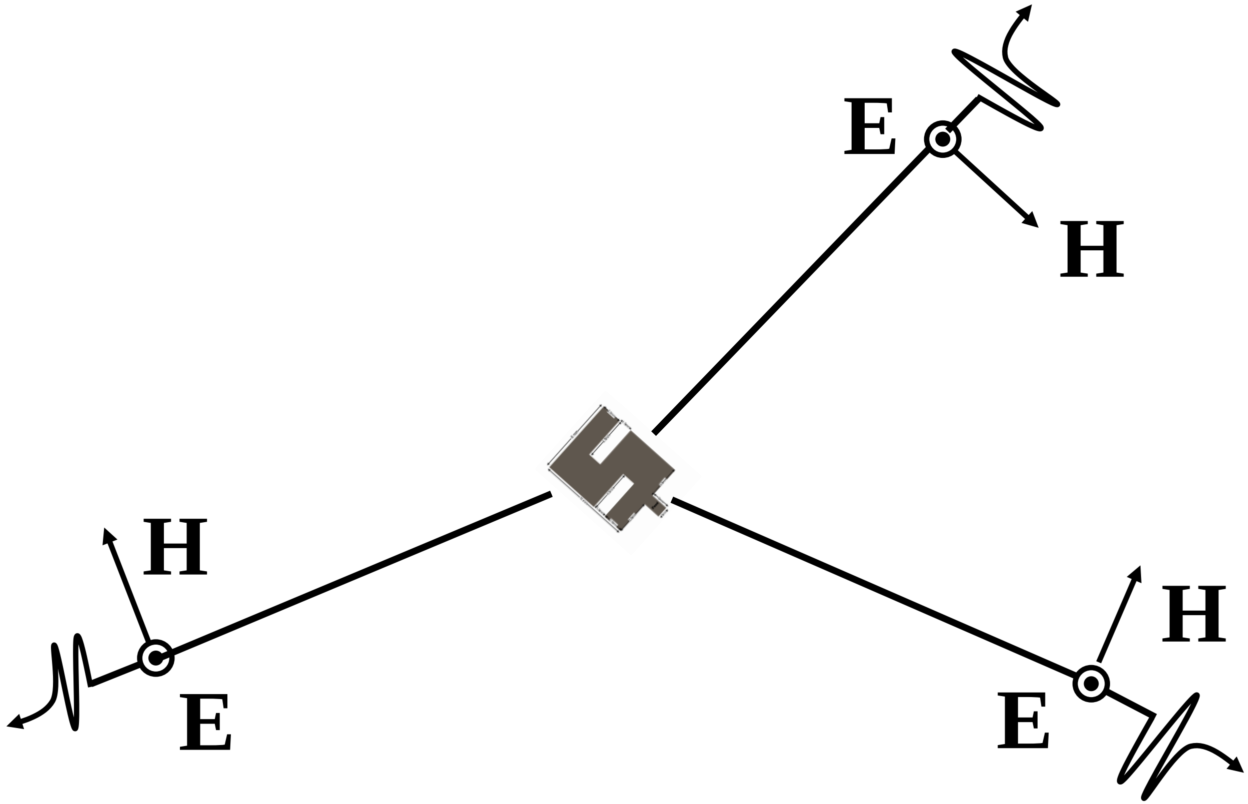
Why Spherical Coordinates?





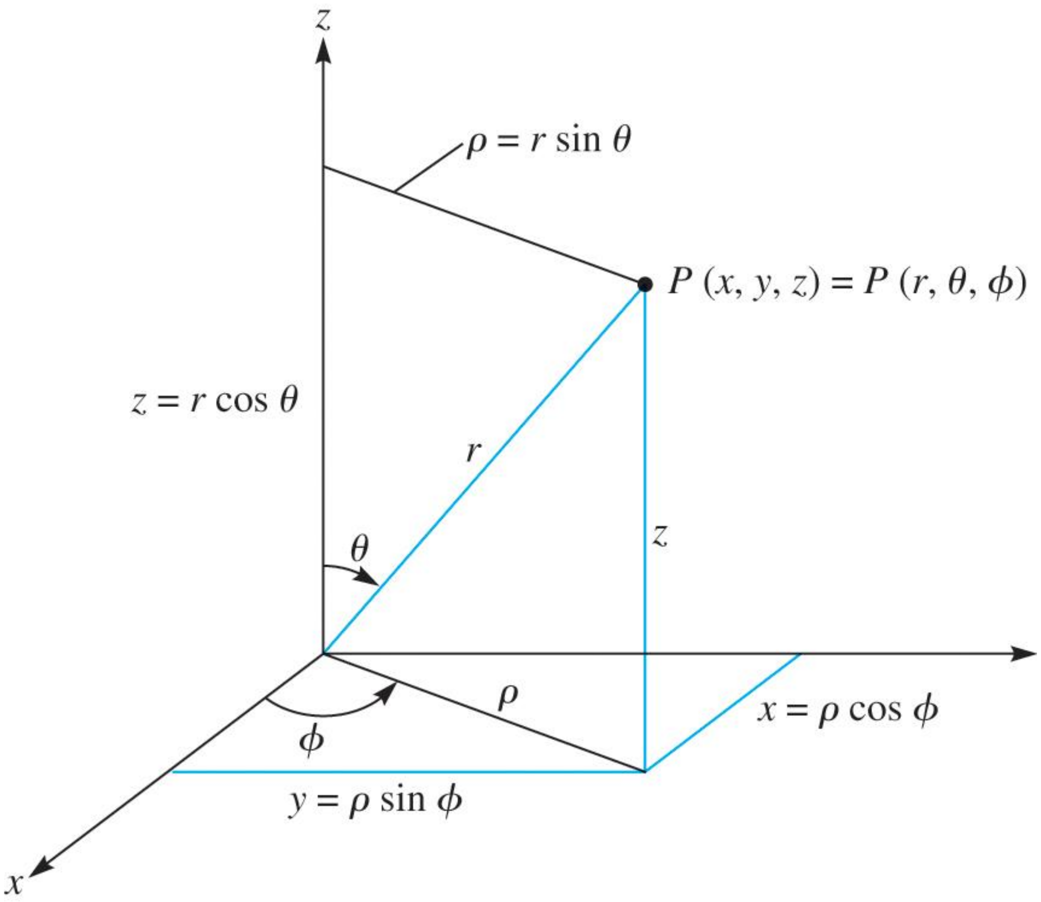
Habib Ullah et al., ACES, 2013

Why Spherical Coordinates? (Cont'd)



The direction of wave propagation, the direction of the electric field, and the direction of the magnetic field form a right handed coordinate system at every point in space!

Coordinate Transformation



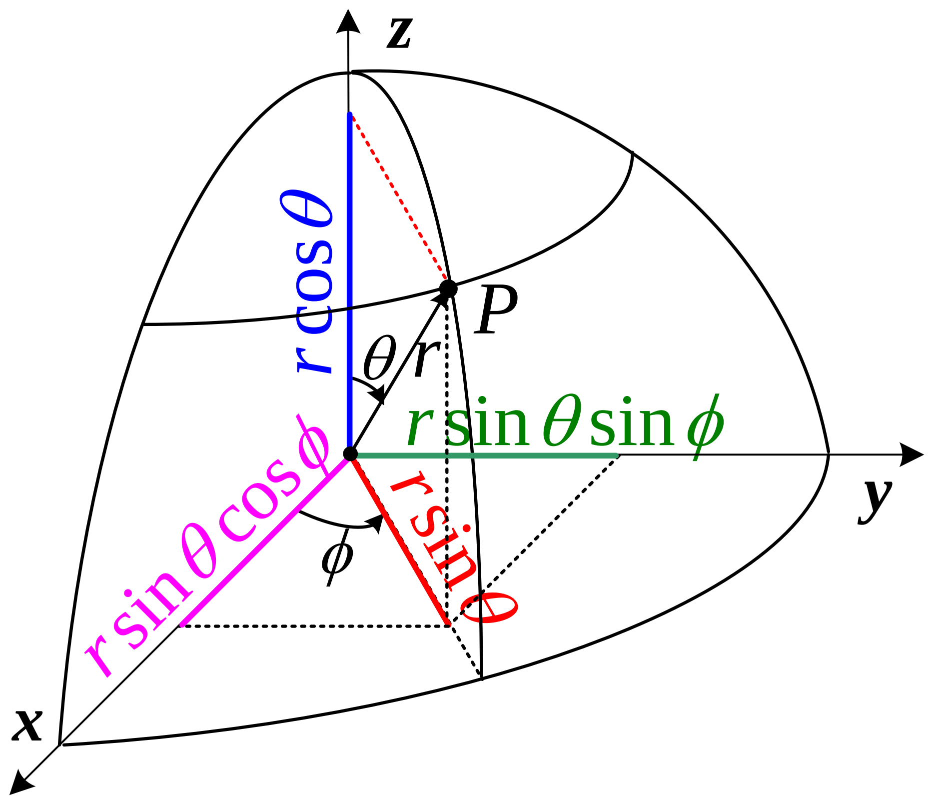
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\phi = \arctan\left(\frac{y}{x}\right)$$

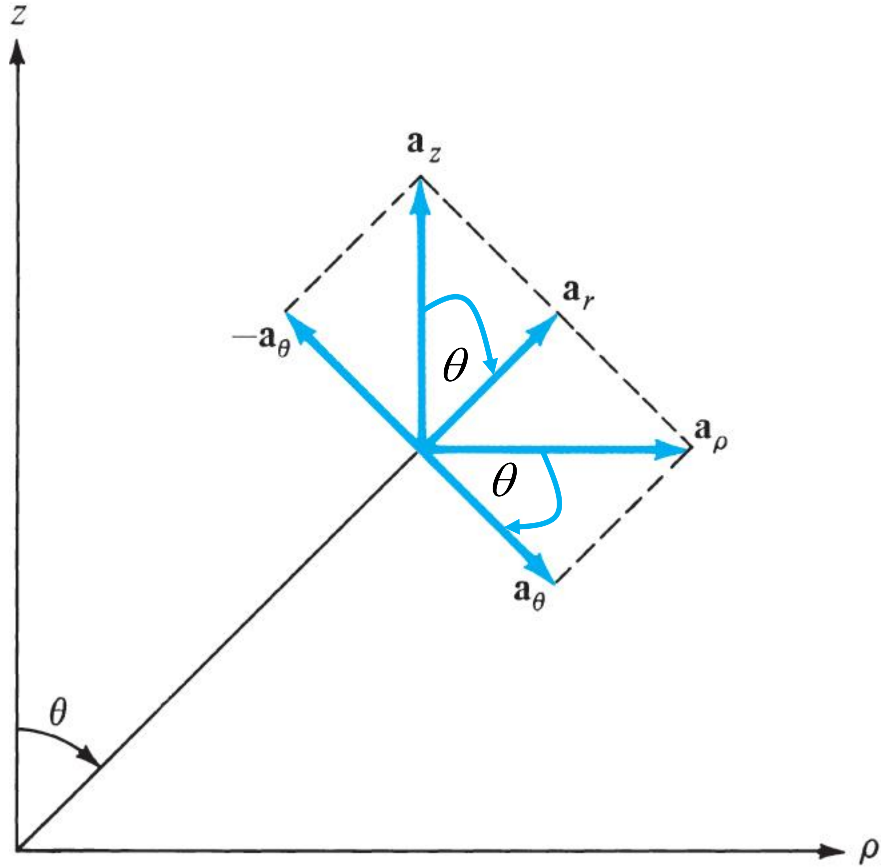
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Coordinate Transformation (Cont'd)



Remember that $\rho = r\ \sin\theta$!

Vector Transformation



$\boldsymbol{a}_r = \sin\theta\, \boldsymbol{a}_\rho + \cos\theta\, \boldsymbol{a}_z$

$\boldsymbol{a}_\theta = \cos\theta\, \boldsymbol{a}_\rho - \sin\theta\, \boldsymbol{a}_z$

Substituting for $\boldsymbol{a}_\rho$ with
$\boldsymbol{a}_\rho = \cos\varphi\, \boldsymbol{a}_x + \sin\varphi\, \boldsymbol{a}_y$,
one gets

$\boldsymbol{a}_r = \sin\theta\cos\varphi\, \boldsymbol{a}_x + \sin\theta\sin\varphi\, \boldsymbol{a}_y + \cos\theta\, \boldsymbol{a}_z$

$\boldsymbol{a}_\theta = \cos\theta\cos\varphi\, \boldsymbol{a}_x + \cos\theta\sin\varphi\, \boldsymbol{a}_y - \sin\theta\, \boldsymbol{a}_z$

$\boldsymbol{a}_\varphi = -\sin\varphi\, \boldsymbol{a}_x + \cos\varphi\, \boldsymbol{a}_y$

Transformation (Cont'd)

If $\mathbf{A}=A_r\,\mathbf{a}_r+A_\theta\mathbf{a}_\theta+A_\varphi\mathbf{a}_\varphi$, we can write

$\mathbf{A}=A_r\,(\sin\theta\cos\varphi\,\boldsymbol{a}_x + \sin\theta\sin\varphi\,\boldsymbol{a}_y + \cos\theta\,\boldsymbol{a}_z)+$

$A_\theta(\cos\theta\cos\varphi\,\boldsymbol{a}_x + \cos\theta\sin\varphi\,\boldsymbol{a}_y - \sin\theta\,\boldsymbol{a}_z) + A_\varphi(-\sin\varphi\,\boldsymbol{a}_x + \cos\varphi\,\boldsymbol{a}_y)$

Collecting terms we have

$\mathbf{A}=(A_r\sin\theta\cos\varphi+A_\theta\cos\theta\cos\varphi - A_\varphi\sin\varphi)\boldsymbol{a}_x+(A_r\sin\theta\sin\varphi + A_\theta\cos\theta\sin\varphi+A_\varphi\cos\varphi)\boldsymbol{a}_y+(A_r\cos\theta - A_\theta\sin\theta)\,\boldsymbol{a}_z$

In matrix form we have

Transformations (Cont'd)

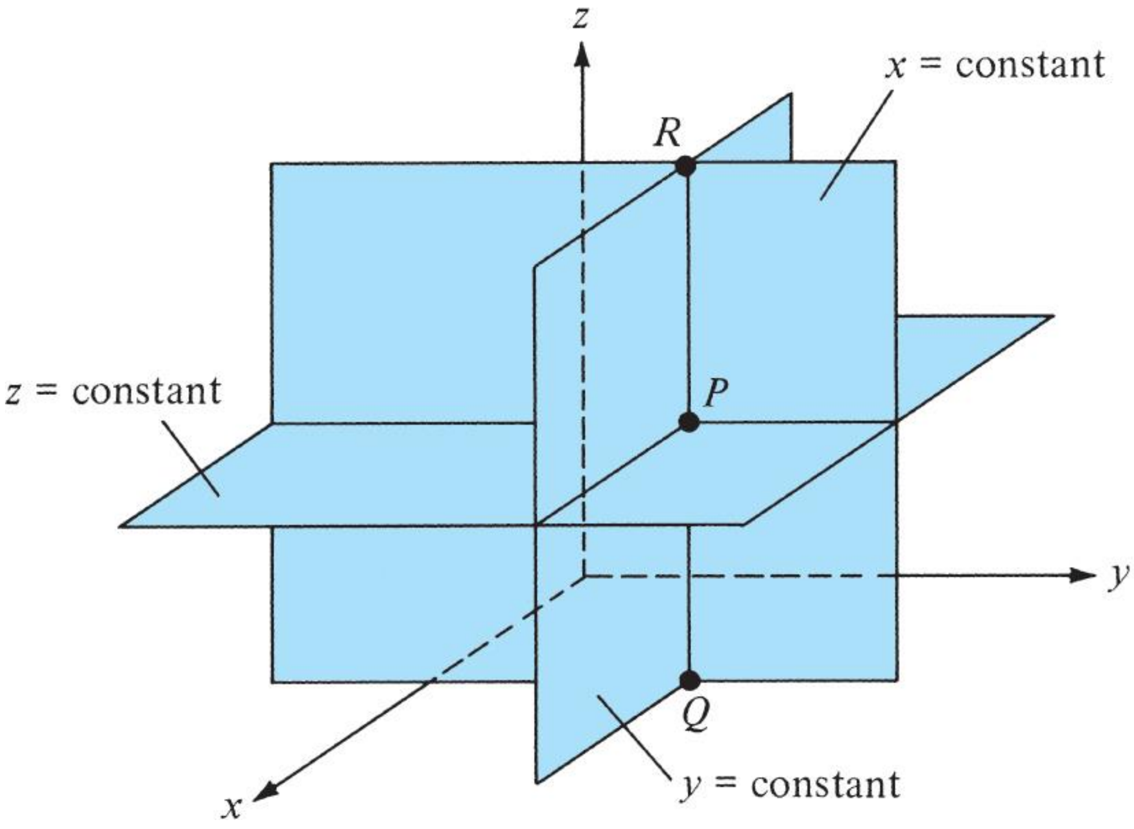
$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix}$$

with the inverse

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

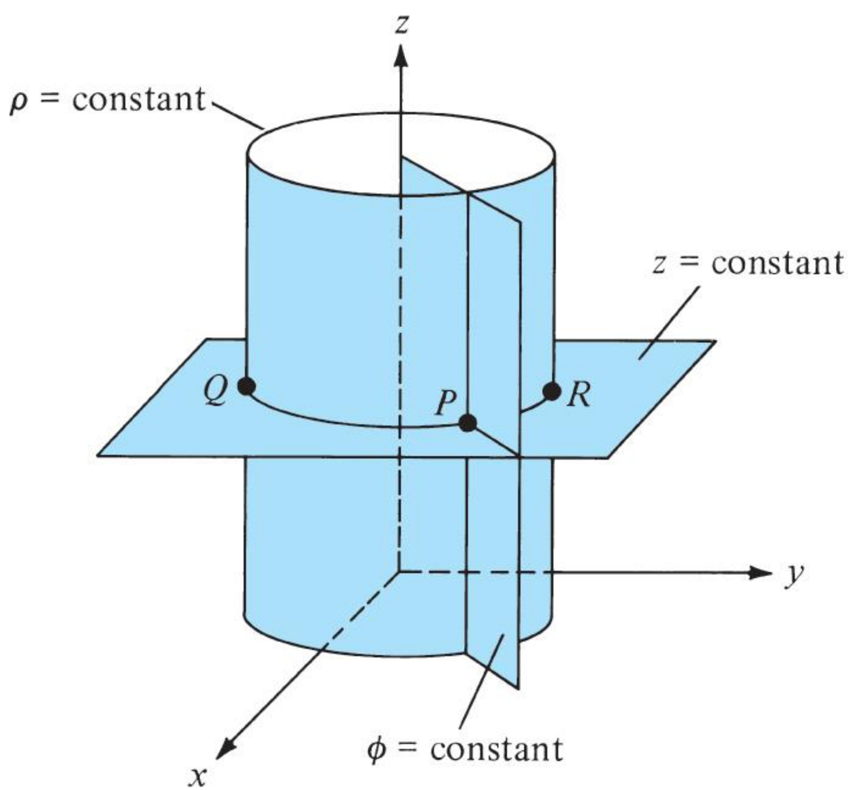
What is the special property of this matrix?

Constant Value Surfaces



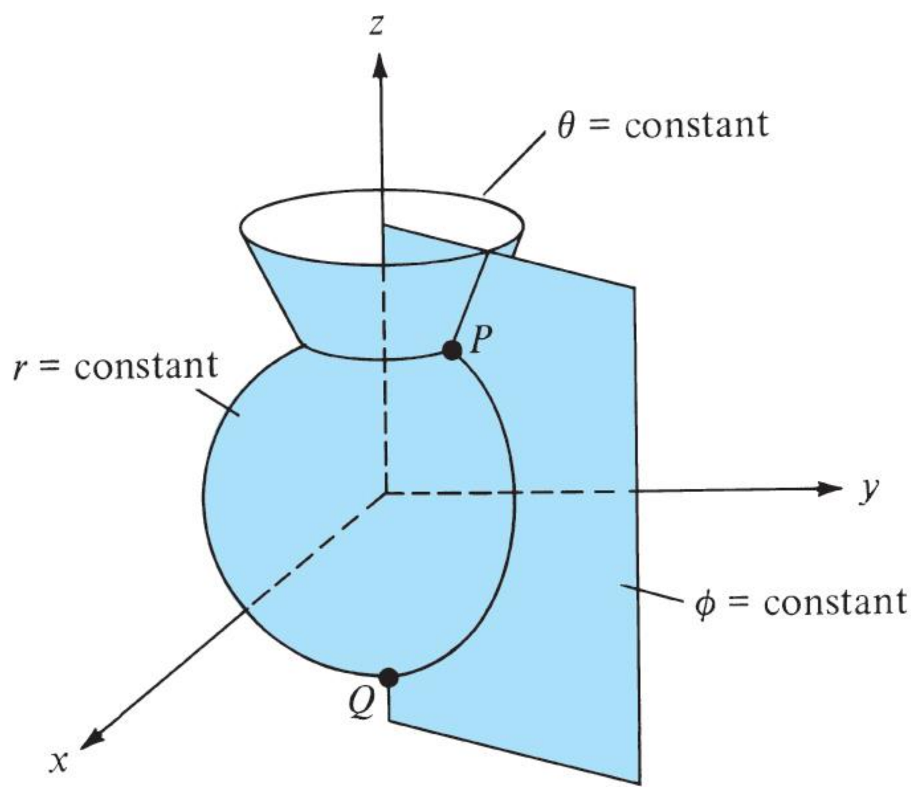
any point P is the intersection of 3 principal planes (constant coordinate surfaces)

Constant Value Surfaces (Cont'd)



Any point in cylindrical coordinates is the intersection of a cylinder, a horizontal plane, and a vertical plane.

Constant Value Surfaces (Cont’d)



In spherical coordinates, a point is the intersection of a cone, a sphere, and a vertical plane