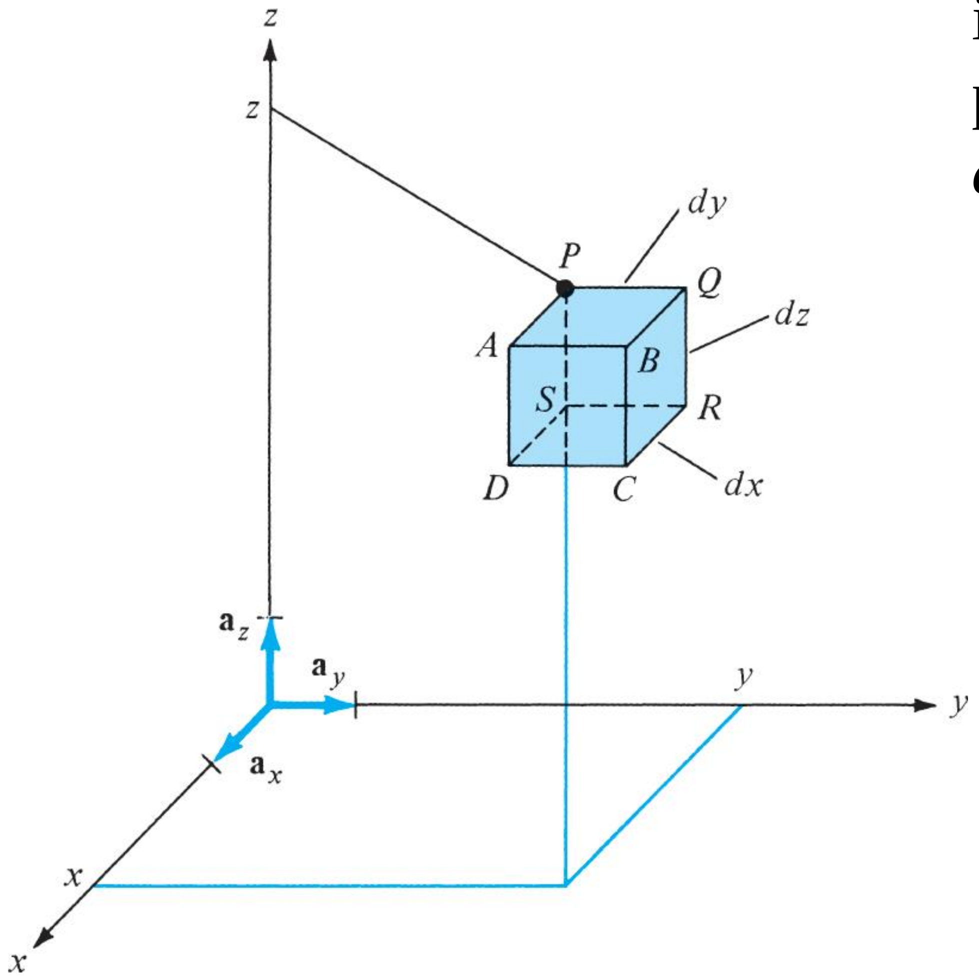
Lecture 4: Vector Calculus

Differential Elements, line integrals, surface integrals, volume integrals

Chapter 3, pages 57-67

Cartesian Differential Element

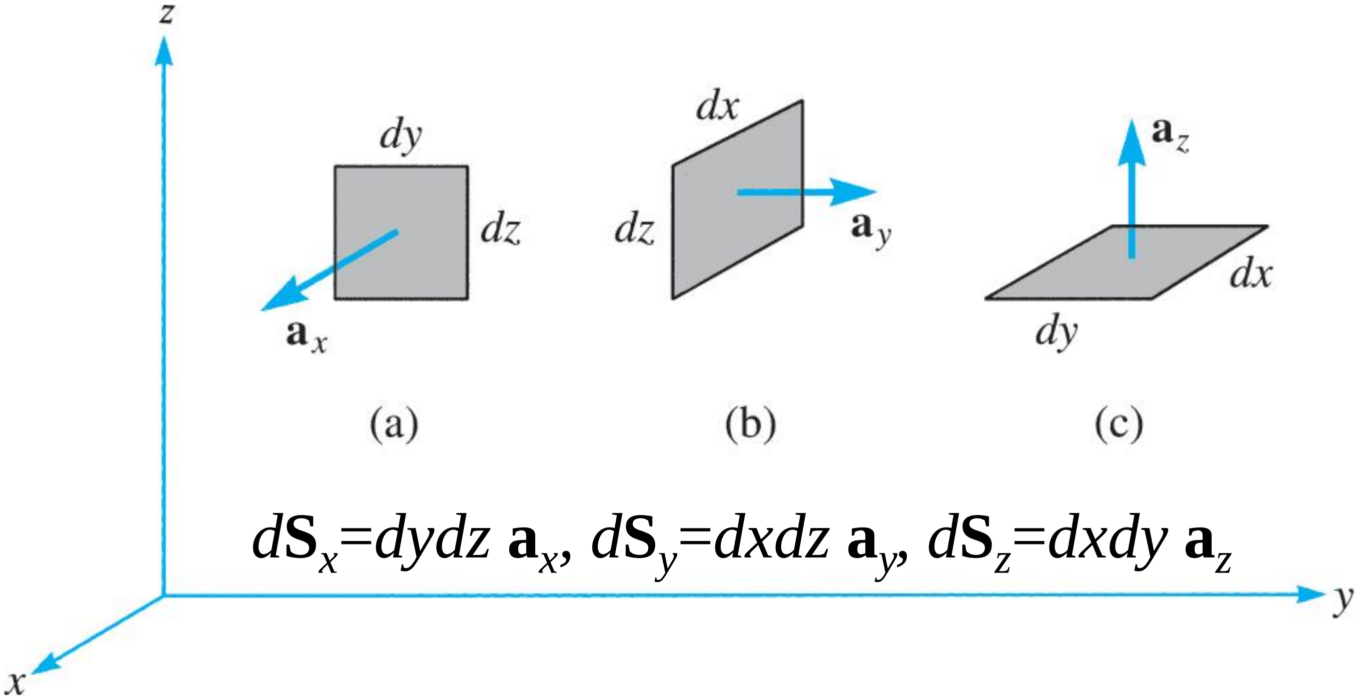


in moving from point S to point B, we increase *x* by *dx*, *y* by *dy*, and *z* by *dz*

the length element directed from S to B is $d\mathbf{l} = dx\, \mathbf{a}_x + dy\, \mathbf{a}_y + dz\, \mathbf{a}_z$

this differential elements defines 6 differential surfaces and one differential volume

Differential Surfaces

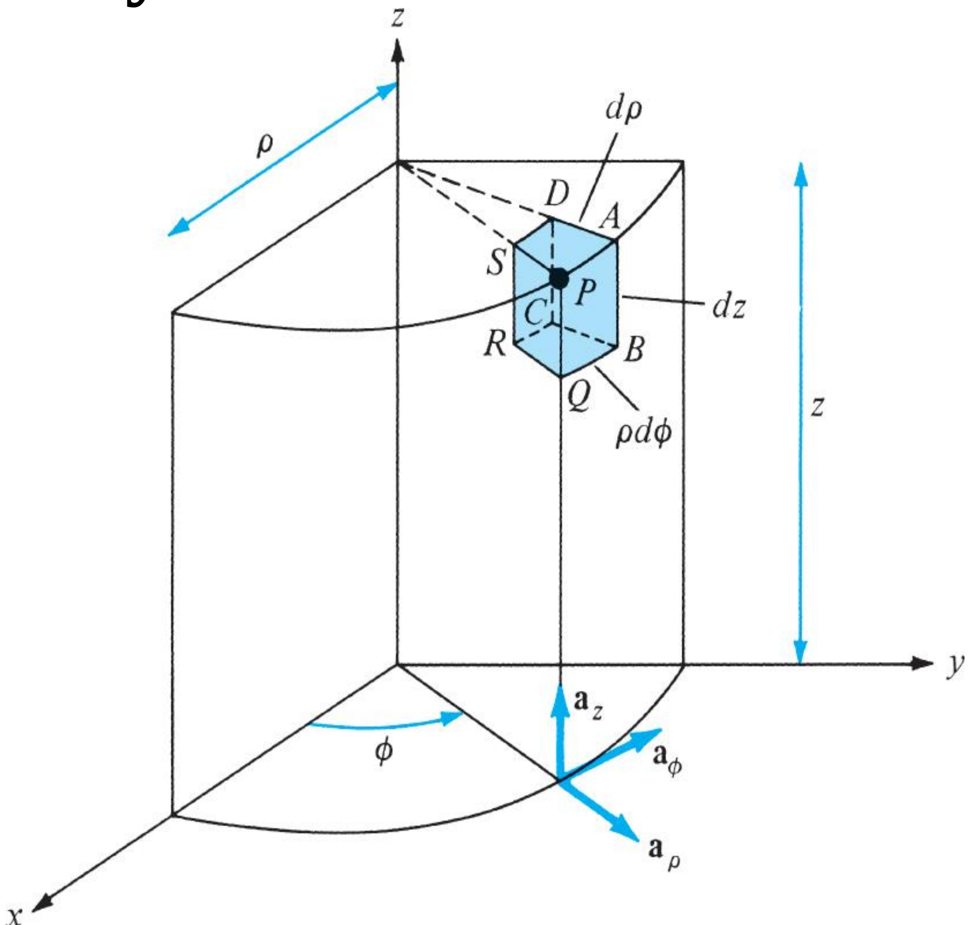


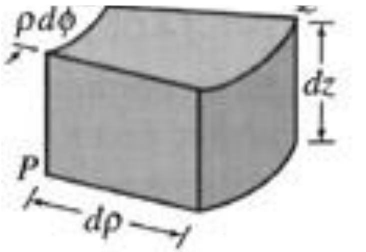
$d\mathbf{S}_x = dydz\ \mathbf{a}_x,\ d\mathbf{S}_y = dxdz\ \mathbf{a}_y,\ d\mathbf{S}_z = dxdy\ \mathbf{a}_z$

notice that there are 3 other differential surfaces with opposite directions

differential volume $dv = dydxdz$

Cylindrical Differential Element



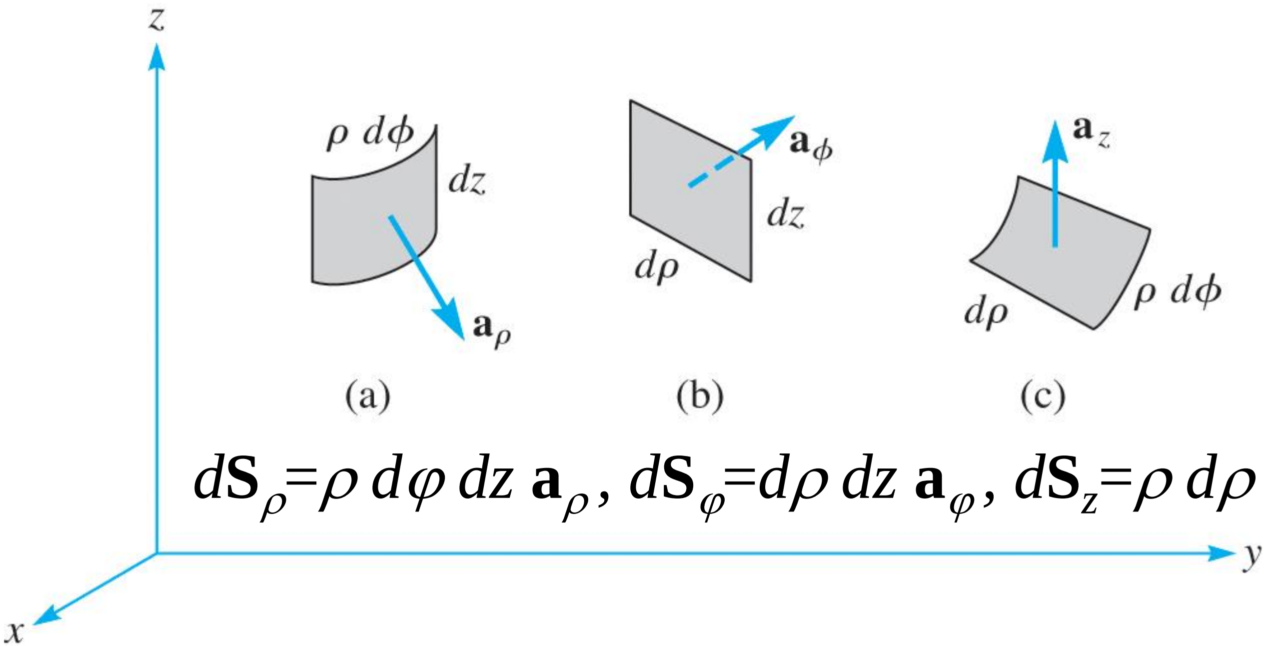


from point R to point A, we perturb ρ by $d\rho$, φ by $d\varphi$, and z by dz

the element has edges of lengths $d\rho$, $\rho\, d\varphi$, and dz:
$d\mathbf{l} = d\rho\, \boldsymbol{a}_\rho + \rho\, d\varphi\, \boldsymbol{a}_\varphi + dz\, \boldsymbol{a}_z$

this differential element defines 6 differential surfaces and one differential volume

Differential Cylindrical Surfaces

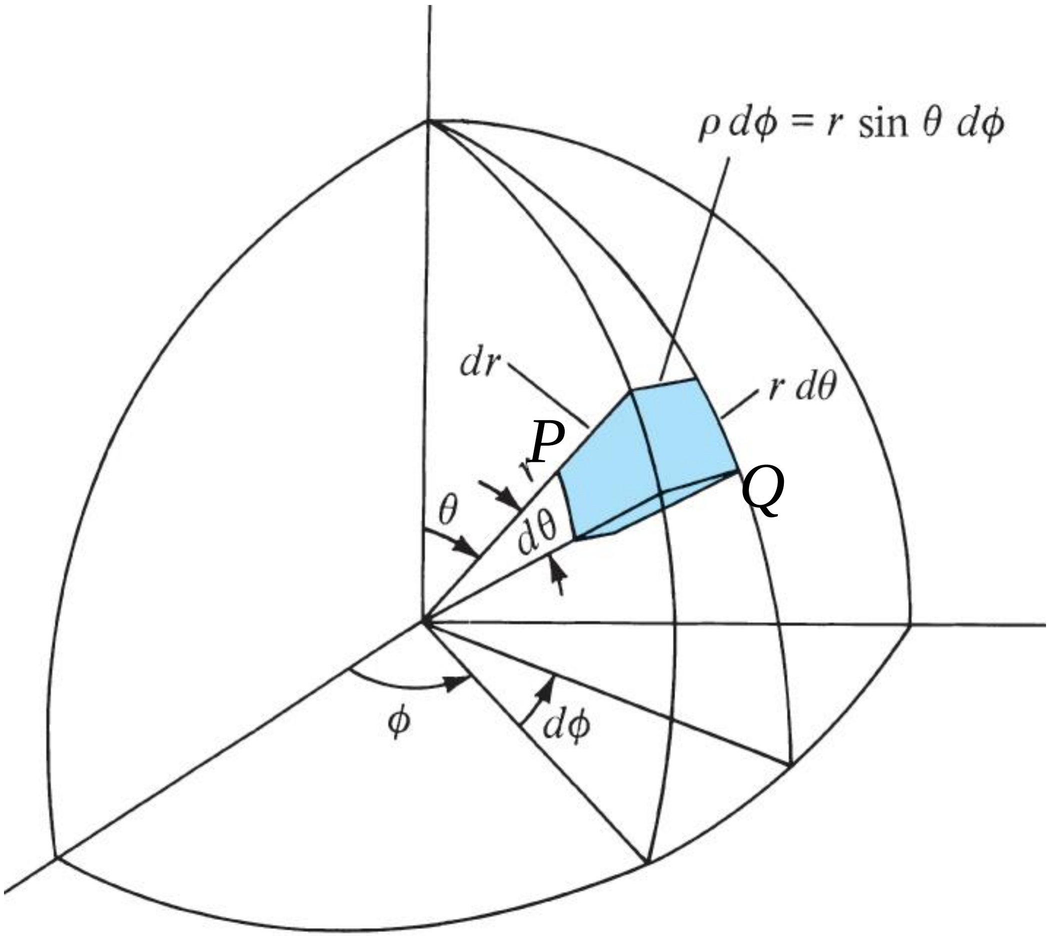


$$d\mathbf{S}_{\rho}=\rho\, d\varphi\, dz\ \mathbf{a}_{\rho},\ d\mathbf{S}_{\varphi}=d\rho\, dz\ \mathbf{a}_{\varphi},\ d\mathbf{S}_{z}=\rho\, d\rho\, d\varphi\ \mathbf{a}_{z}$$

three other differential area vectors are pointing in the opposite directions

differential volume $dv=\rho\, d\rho\, d\varphi\, dz$

Differential Spherical Element



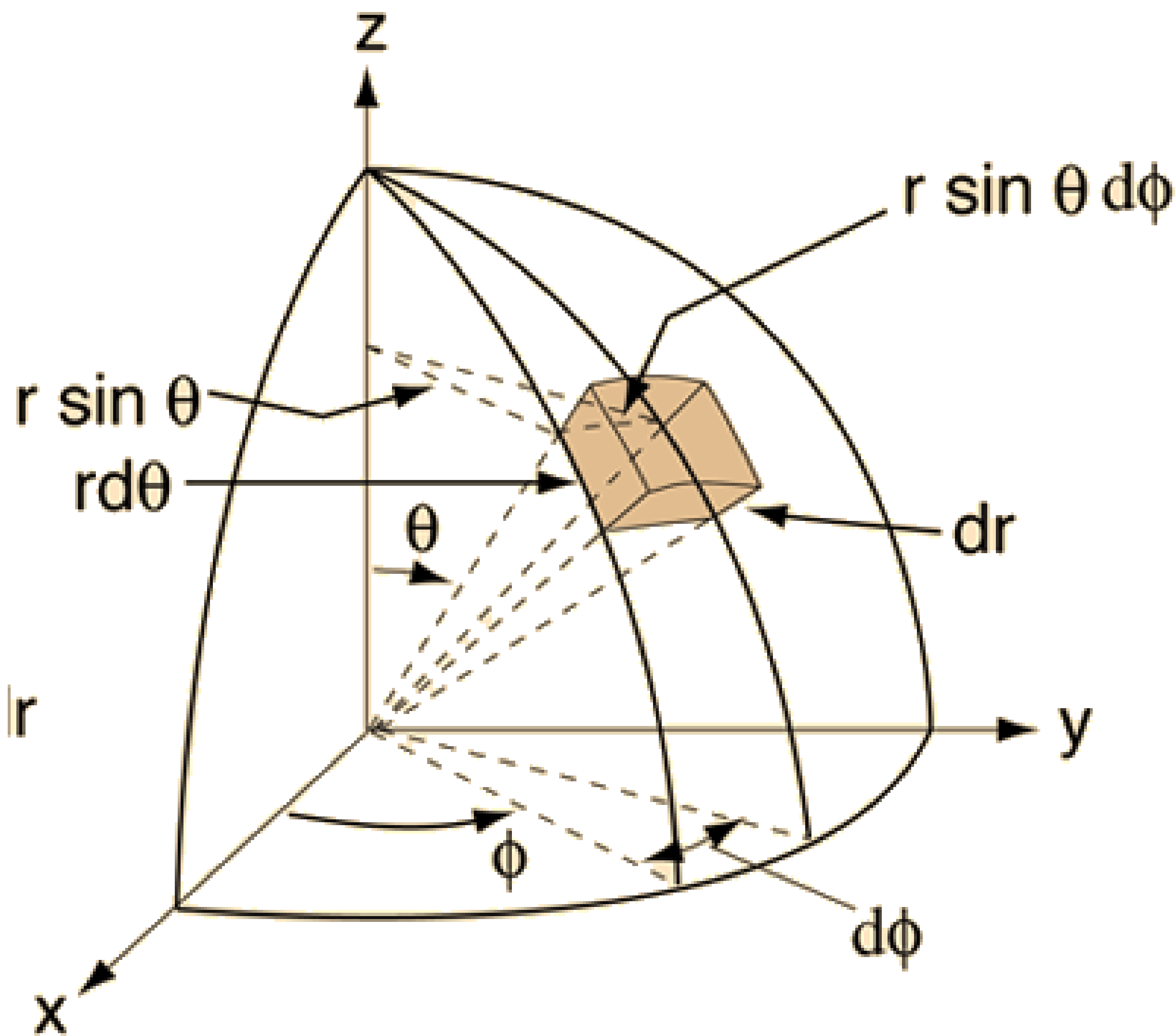
from point P to point Q, we perturb r by dr, θ by $d\theta$, and φ by $d\varphi$

the element has edges of lengths dr, $rd\theta$, and $r\sin\theta\, d\varphi$:

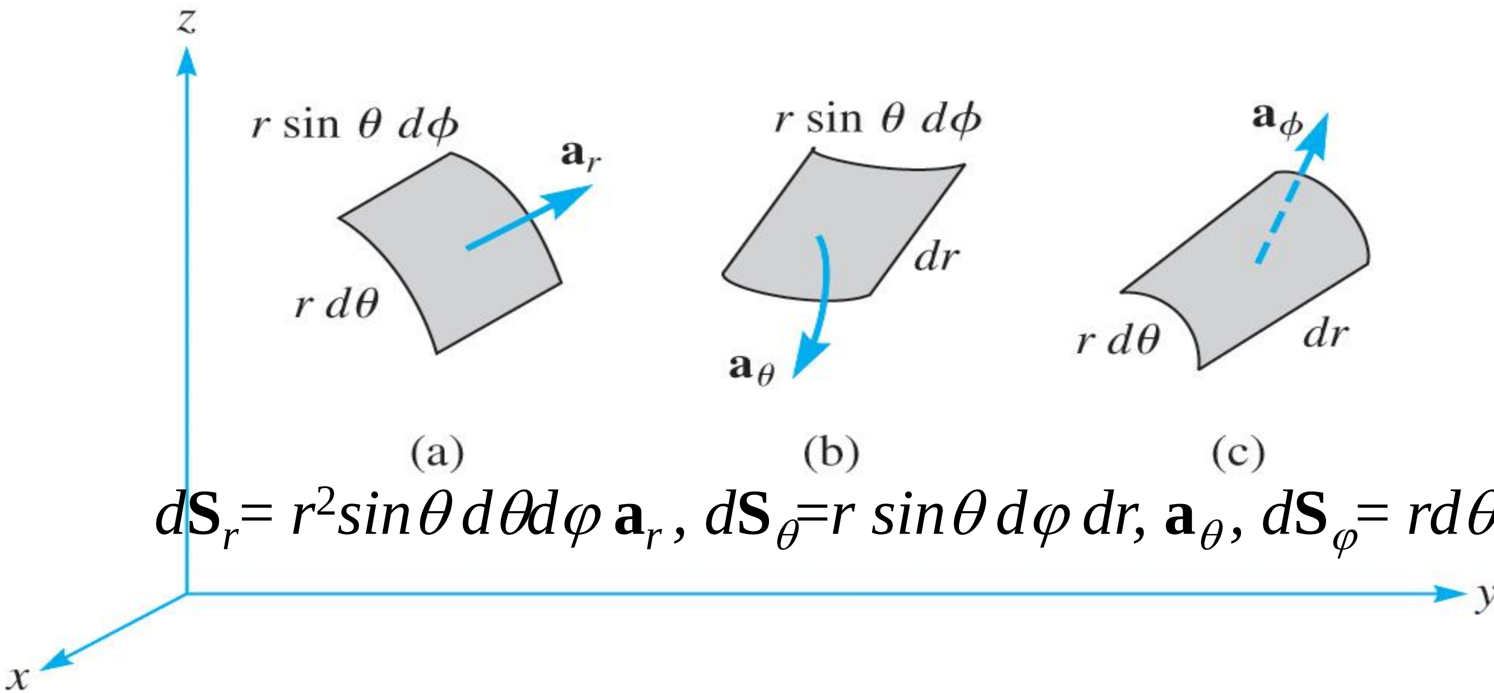
$d\mathbf{l} = dr\, \boldsymbol{a}_r + rd\theta\, \boldsymbol{a}_\theta + r\sin\theta d\varphi\, \boldsymbol{a}_\varphi$

this differential element defines 6 differential surfaces and one differential volume

Differential Spherical Element (Cont'd)



Differential Spherical Surfaces

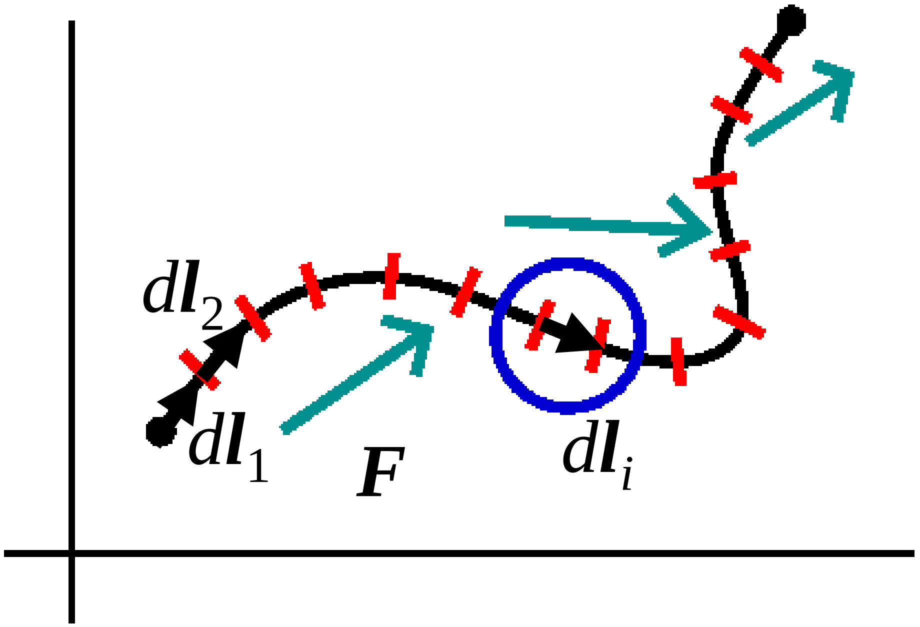


$d\mathbf{S}_r = r^2 sin\theta\, d\theta d\varphi\, \mathbf{a}_r$, $d\mathbf{S}_\theta = r\ sin\theta\, d\varphi\, dr,\ \mathbf{a}_\theta$, $d\mathbf{S}_\varphi = rd\theta dr\ \mathbf{a}_\varphi$

three other surface elements are points in the opposite directions

the differential volume is $dv = r^2 sin\theta\, drd\theta d\varphi$

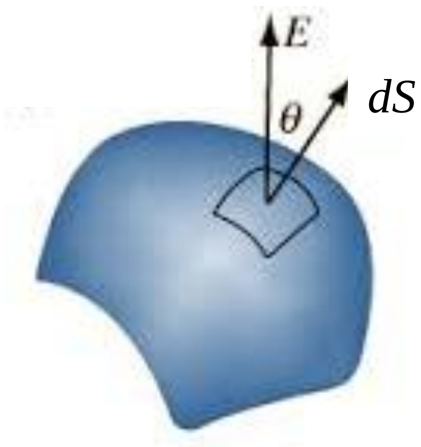
Motivation for Line Integrals

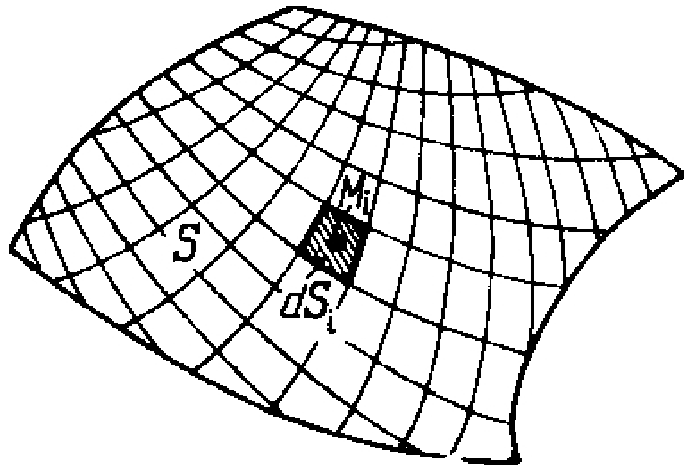


work done $W = \sum_{i=1}^{n} \boldsymbol{F} . d\boldsymbol{l}_i = \sum_{i=1}^{n} F_i . dl_i \cos\theta_{\boldsymbol{i}}$

as $n \to \infty$, summation becomes a line integral $W = \int \boldsymbol{F} . d\boldsymbol{l}$

Motivation for Surface Integrals



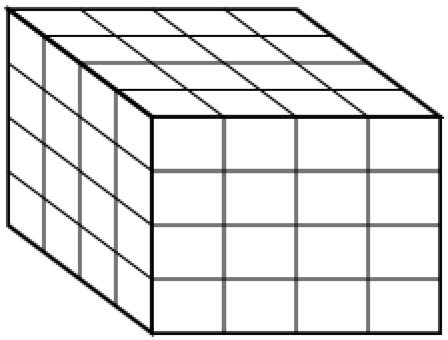


the flux of a vector through a small surface element is the product of the **normal component** of the vector with the area of that element $\psi_i = E_i \cos\theta_i \, dS_i$

Total flux over a surface $= \sum_{i=1}^{n} \psi_i = \sum_{i=1}^{n} E_i \, dS_i \cos\theta_i = \sum_{i=1}^{n} \boldsymbol{E}_i \bullet d\boldsymbol{S}_i$

as $n \to \infty$, summation becomes a surface integral $\psi = \iint \boldsymbol{E} \bullet d\boldsymbol{S}$

Motivation for Volume Integrals



in the general case, the density of an object is not uniform

the mass of a differential volume element is given by $dm_i = \rho_i dv_i$

the total mass is the sum of all differential masses $m = \sum_{i=1}^{n} \rho_i \, dv_i$

as $n \rightarrow \infty$, summation becomes a volume integral $m = \iiint \rho dv$

dv can be a differential volume in any of the three coordinate systems