Lecture 5: Vector Calculus

Del operator, gradient of a scalar, Divergence operator, Divergence theorem, Chapter 3, pages 67-79

Del Operator

the del operator is a mathematical operator used very often in different branches of science

in Cartesian coordinates $\nabla = \boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z}$

this operator can be used in 3 different ways:

gradient: scalar to vector $\boldsymbol{E} = \nabla V = \boldsymbol{a}_x \frac{\partial V}{\partial x} + \boldsymbol{a}_y \frac{\partial V}{\partial y} + \boldsymbol{a}_z \frac{\partial V}{\partial z}$

divergence: vector to scalar $\nabla \bullet \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$

Curl: vector to vector: $\nabla \times \boldsymbol{E} = \begin{bmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix}$

Del Operator in other OC Systems

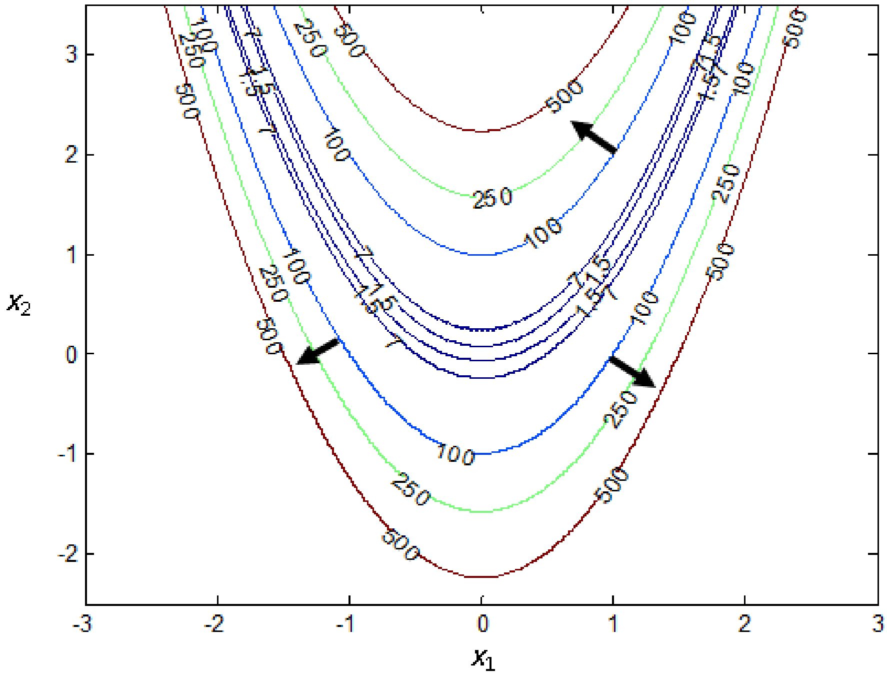
the del operator can also be defined for cylindrical and spherical systems

Cylindrical coordinates $\nabla = \boldsymbol{a}_\rho \frac{\partial}{\partial \rho} + \boldsymbol{a}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \boldsymbol{a}_z \frac{\partial}{\partial z}$

Spherical coordinates $\nabla = \boldsymbol{a}_r \frac{\partial}{\partial r} + \boldsymbol{a}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \boldsymbol{a}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$

to derive these expressions, use chain rule to transform the derivatives and then transform the unit vectors as well.

Gradient



Bakr, IET Press, 2013

$$\nabla V = \boldsymbol{a}_x \frac{\partial V}{\partial x} + \boldsymbol{a}_y \frac{\partial V}{\partial y} + \boldsymbol{a}_z \frac{\partial V}{\partial z}$$

the gradient is a vector normal to the constant value surfaces

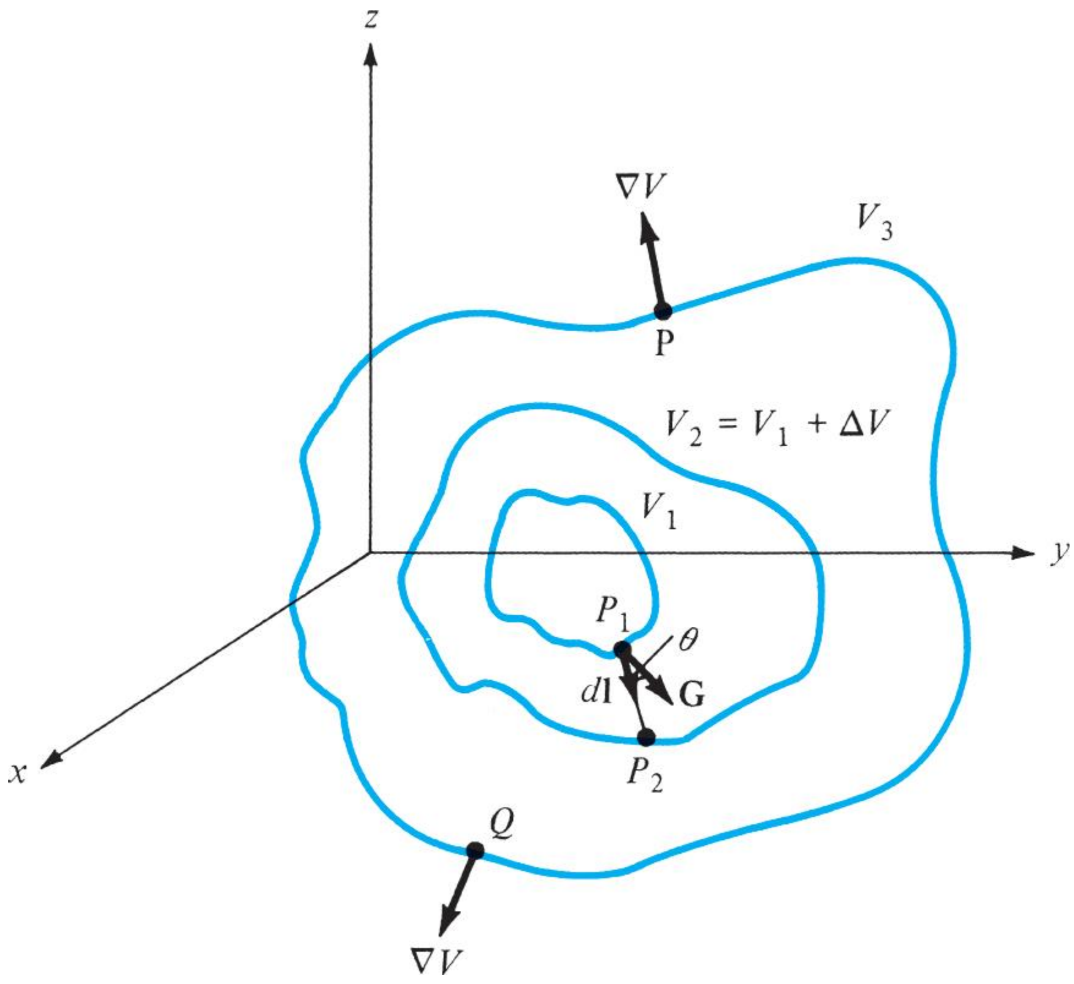
the gradient points in the direction of maximum function increase

for functions of two variables, the constant value surface is a contour

the directional derivative along a direction ***s*** is

$$\nabla V \bullet \boldsymbol{s} = |\nabla V||\boldsymbol{s}|\cos\theta$$

Gradient (Cont'd)



for functions of position, the constant value surfaces are 3D surfaces

the static electric field points in the direction of the negative of the gradient of the electric potential!

Divergence

the divergence of a vector at a point is defined as

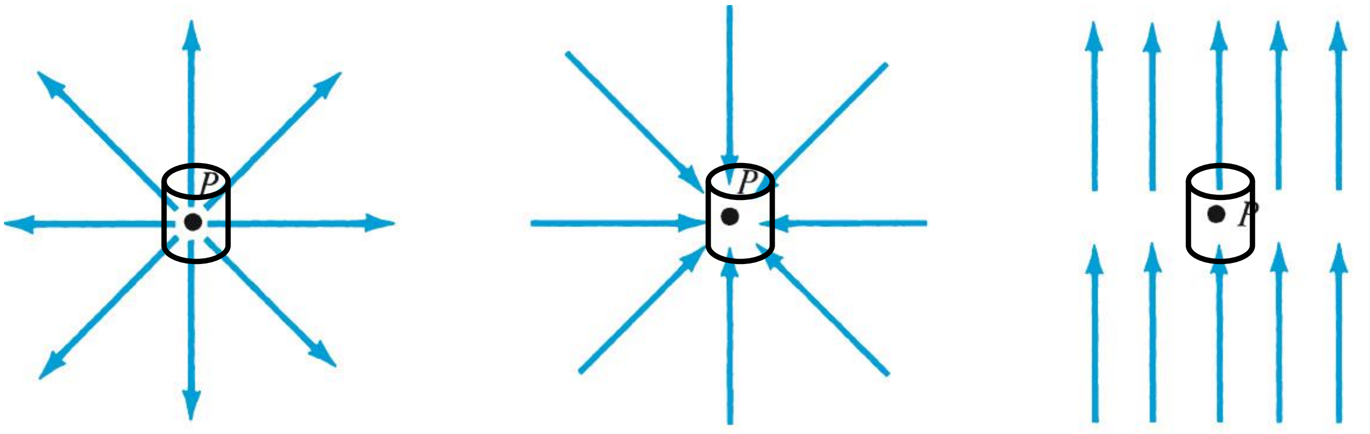
$$\text{div}\,\boldsymbol{F} = \lim_{\Delta V \to 0} \frac{\displaystyle\oint\!\!\!\oint_{\Delta S} \boldsymbol{F}.\boldsymbol{dS}}{\Delta V} = \nabla.\boldsymbol{F}$$

using the differential volume elements, the expressions of divergence in Cartesian, Cylindrical, and Spherical coordinates can be derived.

the divergence of a vector is a scalar value that is position dependent

divergence measures the existence of sources at a point

Divergence Cases

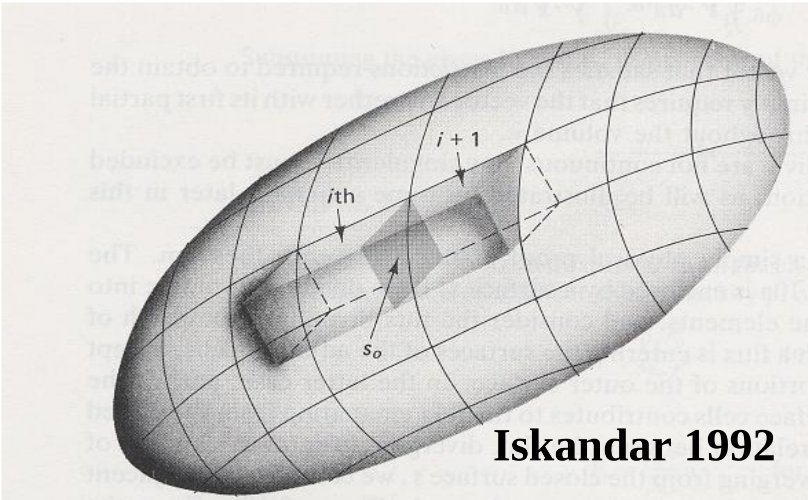


(a) (b) (c)

positive divergence negative divergence zero divergence

static electric field lines diverge from positive charges and converge to negative charges!

Divergence Theorem



Iskandar 1992

Divergence Theorem converts a closed surface integral to a volume integral over the enclosed volume

for the *i*th element we have

$$\oint\!\!\!\oint_{\Delta S_i} \boldsymbol{F.dS} = \text{div}\ \boldsymbol{F}\ \Delta V_i$$

summing over the *N* elements we get $\displaystyle\sum_{i=1}^{N} \oint\!\!\!\oint_{\Delta S_i} \boldsymbol{F.dS} = \sum_{i=1}^{N} \text{div}\ \boldsymbol{F}\ \Delta V_i$

notice that the flux cancels out between adjacent elements leaving only external surface flux

as $N \rightarrow \infty$, we get $\displaystyle\oint\!\!\!\oint_{S} \boldsymbol{F.dS} = \iiint_{V} \nabla.\boldsymbol{F}\ dV$