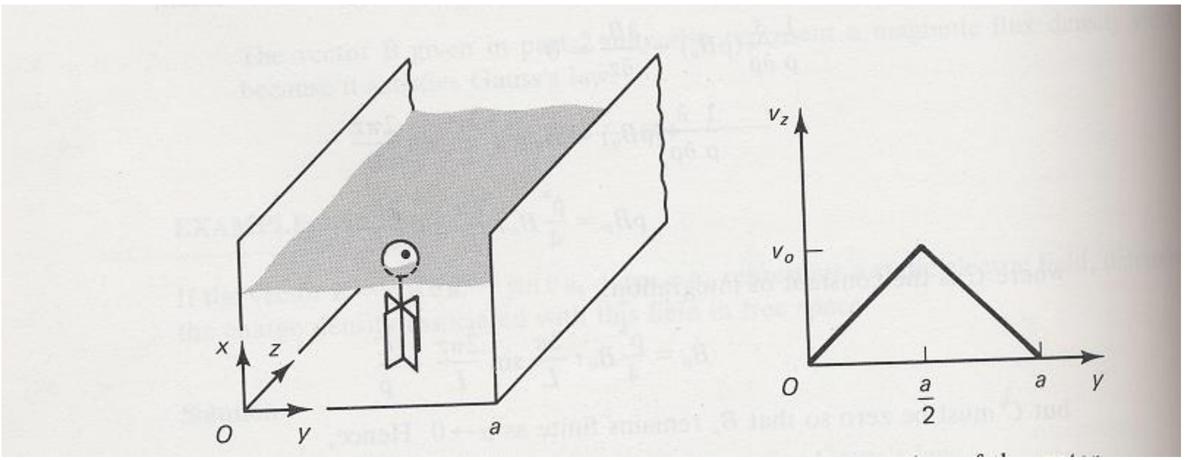
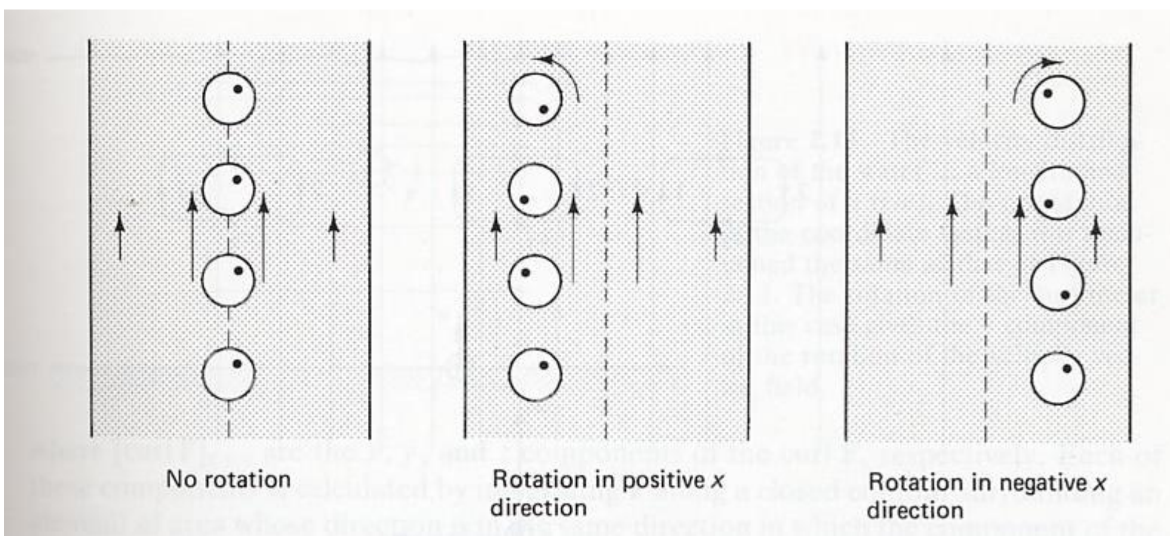
Lecture 6: Vector Calculus

Curl Operator, Stokes Theorem, Classification of Vectors, Laplacian of a scalar, Chapter 3, pages 80-95

Motivation for Curl Operator





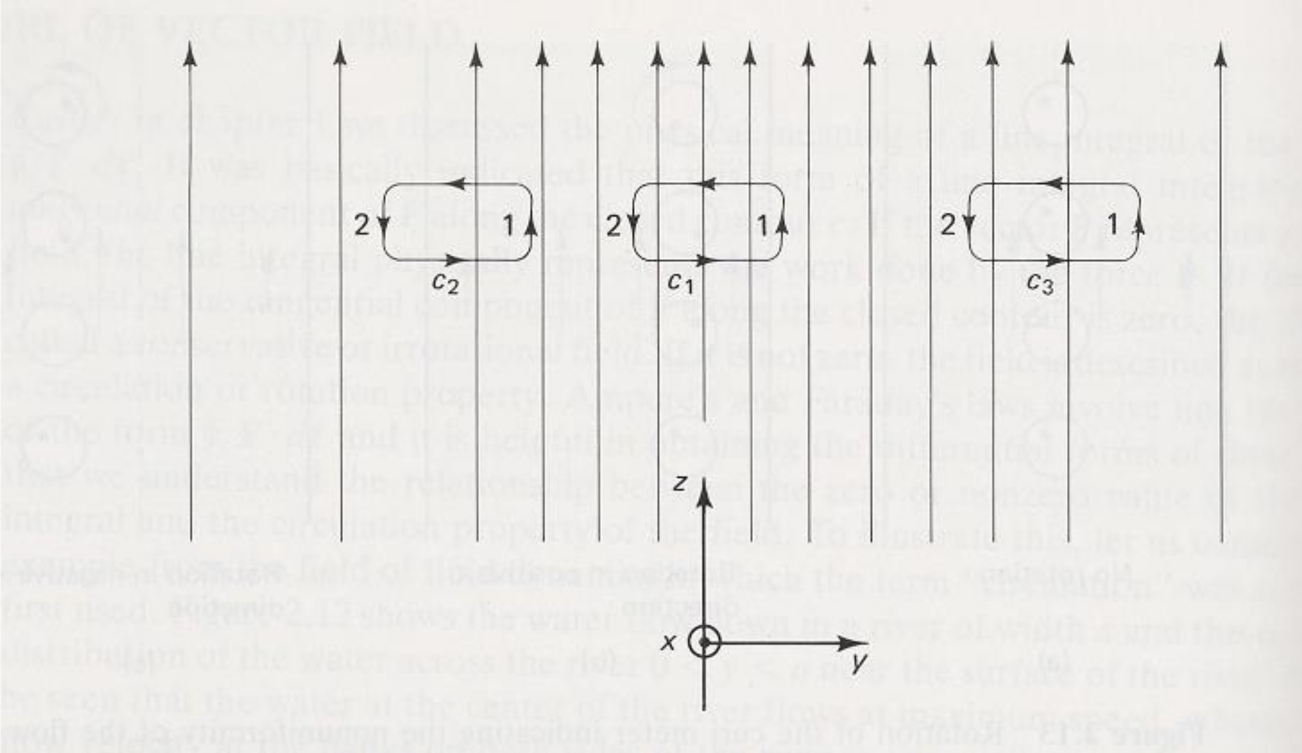
For a mechanical wave, a curlometer can be used

The direction or rotation changes depending on the position within the stream

Iskandar 1992

Motivation (Cont'd)

the curl can be measured through a line integral

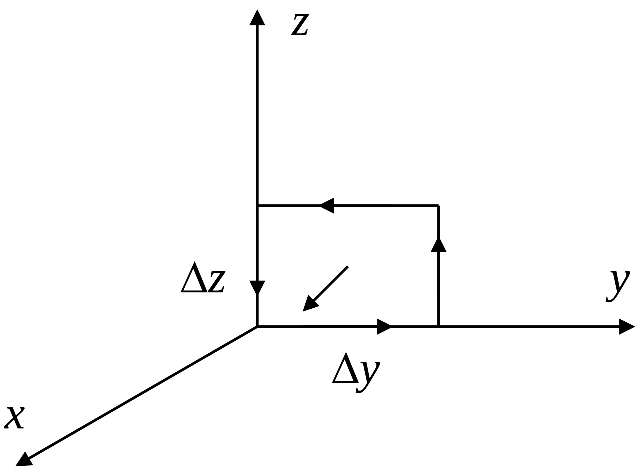


Iskandar, 1992

$$\oint \mathbf{F.dl} = +\text{ve} \qquad \oint \mathbf{F.dl} = 0 \qquad \oint \mathbf{F.dl} = -\text{ve}$$

Curl in Cartesian Coordinates

the curl is a vector normal to the plane of the line integral



$$[\nabla\times\mathbf{F}]_x = \lim_{\Delta y,\Delta z\to 0}\frac{\oint_{C_1}\mathbf{F.dl}}{\Delta y\Delta z}$$

The same approach can be repeated in the *xz*, and *xy* planes to get the other two components of the curl operator

$$\nabla\times\mathbf{A} = \begin{bmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{A}_x & \mathrm{A}_y & \mathrm{A}_z \end{bmatrix}$$

Curl Expressions

$$\nabla \times \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

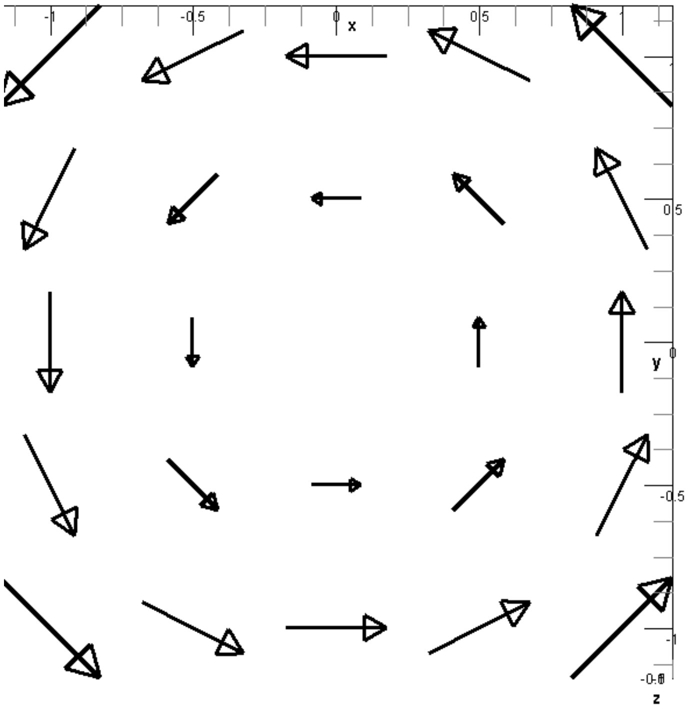
$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \mathbf{a}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \mathbf{a}_\phi + \frac{1}{\rho} \left[\frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right] \mathbf{a}_z$$

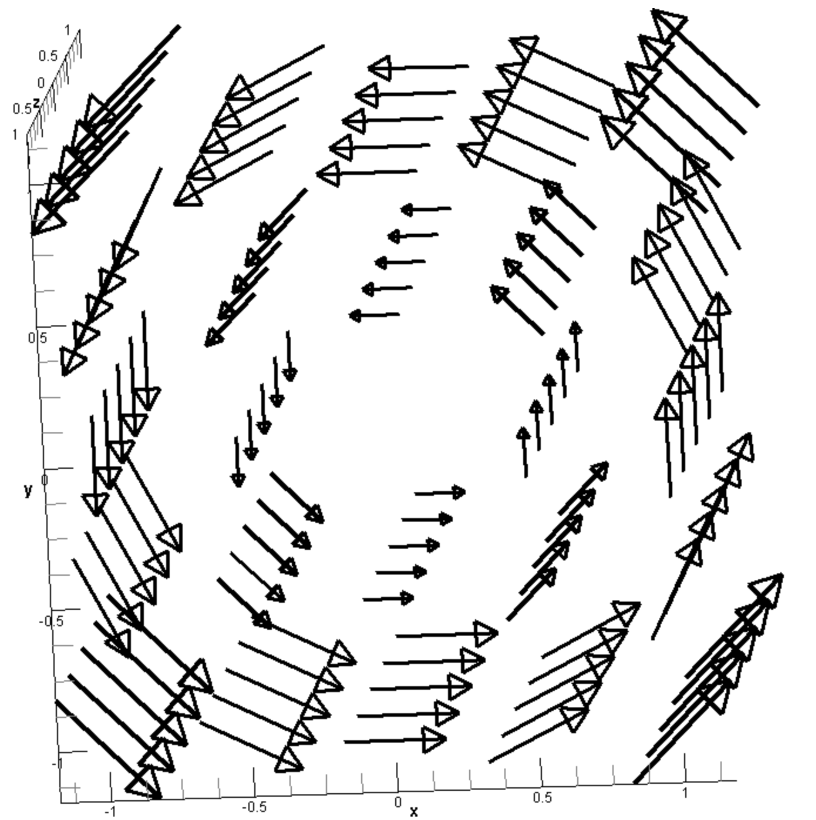
$$(\nabla \times \mathbf{A})_r = \frac{1}{r \sin\theta} \left[\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right]$$

$$(\nabla \times \mathbf{A})_\theta = \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial (r A_\phi)}{\partial r} \right]$$

$$(\nabla \times \mathbf{A})_\phi = \frac{1}{r} \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right]$$

Illustrations of Rotational Vectors

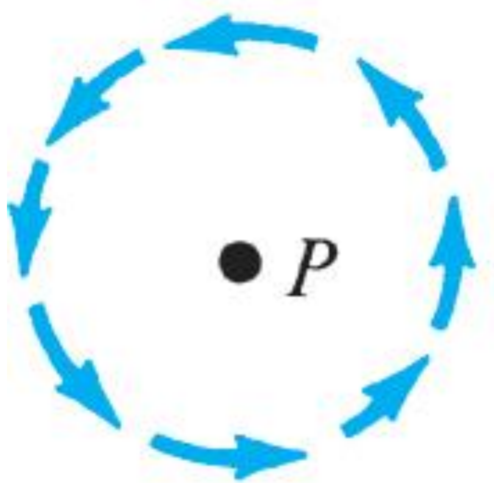




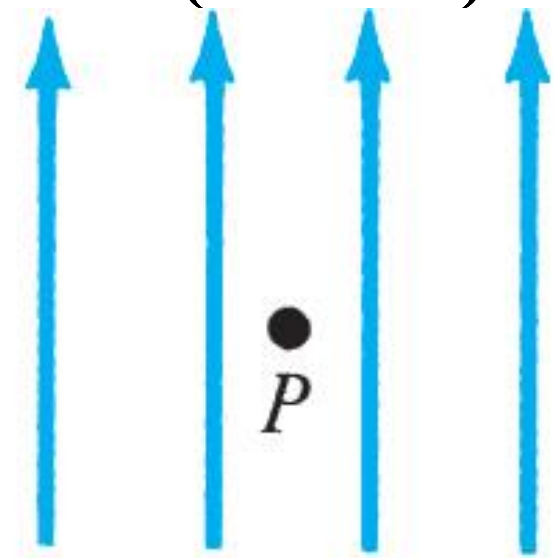
$$\boxed{\mathbf{F} = -y\mathbf{a}_x + x\mathbf{a}_y \Rightarrow \nabla \times \mathbf{F} = 2\mathbf{a}_z}$$

the direction of curl**F** is along the axis of rotation (right-hand rule)

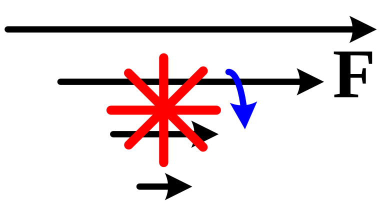
Rotational and Irrotational vectors (Cont'd)



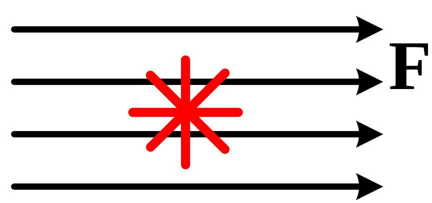
$$\nabla \times \mathbf{A} \neq \mathbf{0}$$



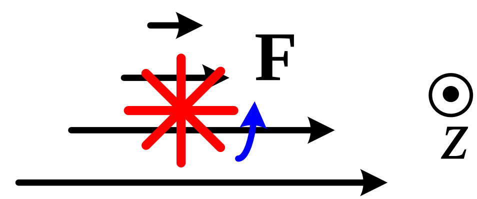
$$\nabla \times \mathbf{A} = \mathbf{0}$$



$$(\nabla \times \mathbf{F})_z < 0$$

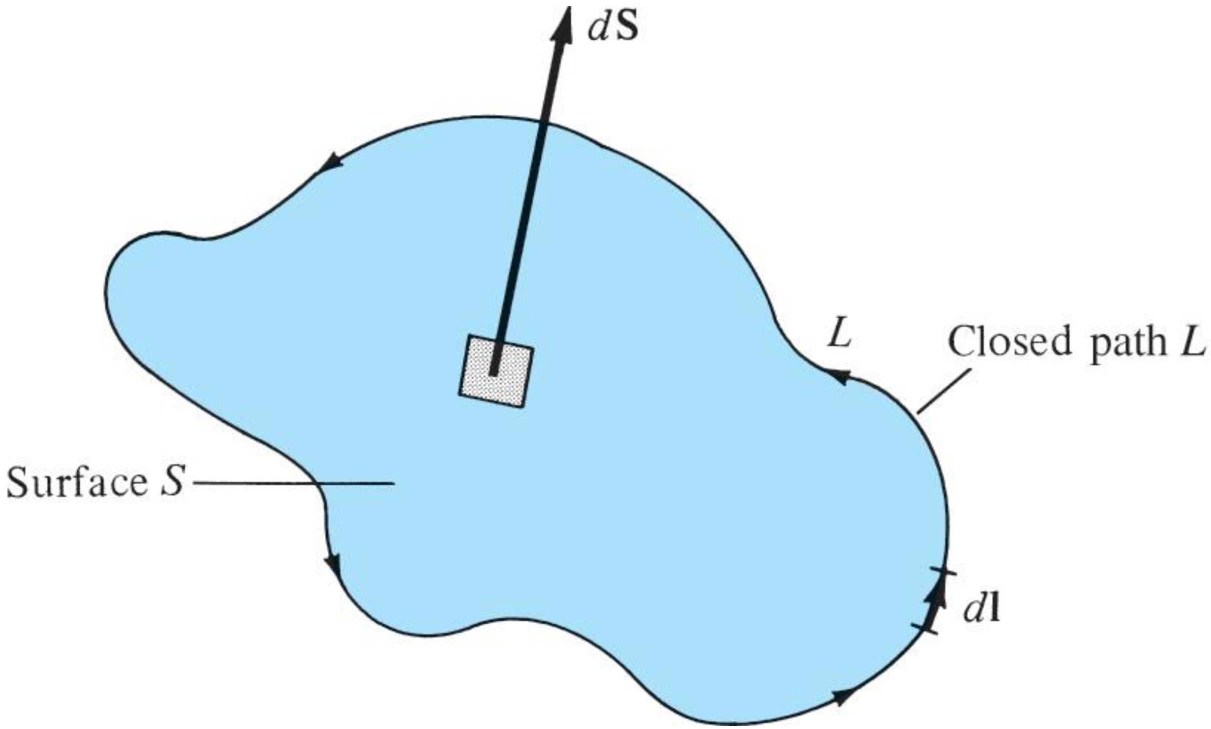


$$(\nabla \times \mathbf{F})_z = 0$$



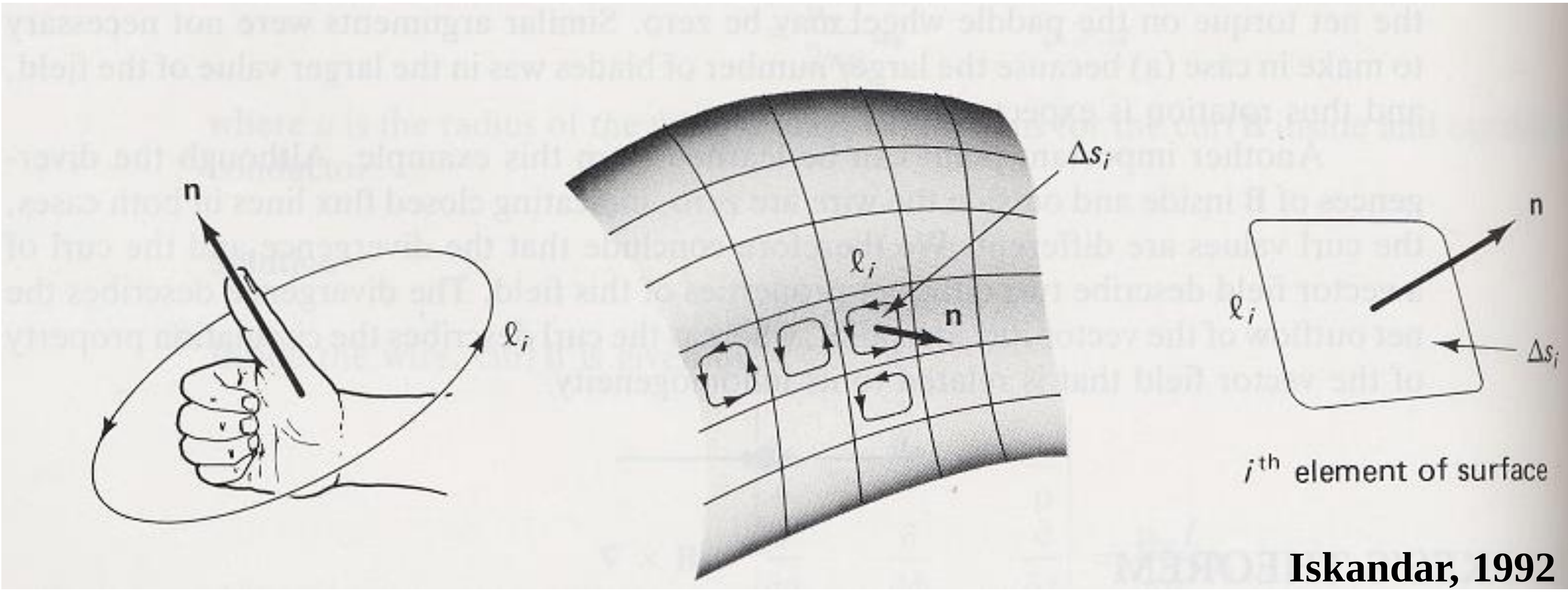
$$(\nabla \times \mathbf{F})_z > 0$$

Stokes Theorem



$$\boxed{\oint_C \mathbf{H}\cdot d\mathbf{L} = \iint_S (\nabla \times \mathbf{H})\cdot d\mathbf{S}}$$

Stokes Theorem (Cont'd)



Iskandar, 1992

- For the *i*th element we have $$\lim_{\Delta S_i \to 0} \frac{\oint_{\ell_i} \mathbf{F}.\mathbf{dl}}{\Delta S_i} = \text{curl } \mathbf{F}.\mathbf{n}$$

or $$\oint_{\ell_i} \mathbf{F}.\mathbf{dl} = \text{curl } \mathbf{F}.\Delta\mathbf{S}_i$$

Stokes Theorem (Cont'd)

summing for all elements we get

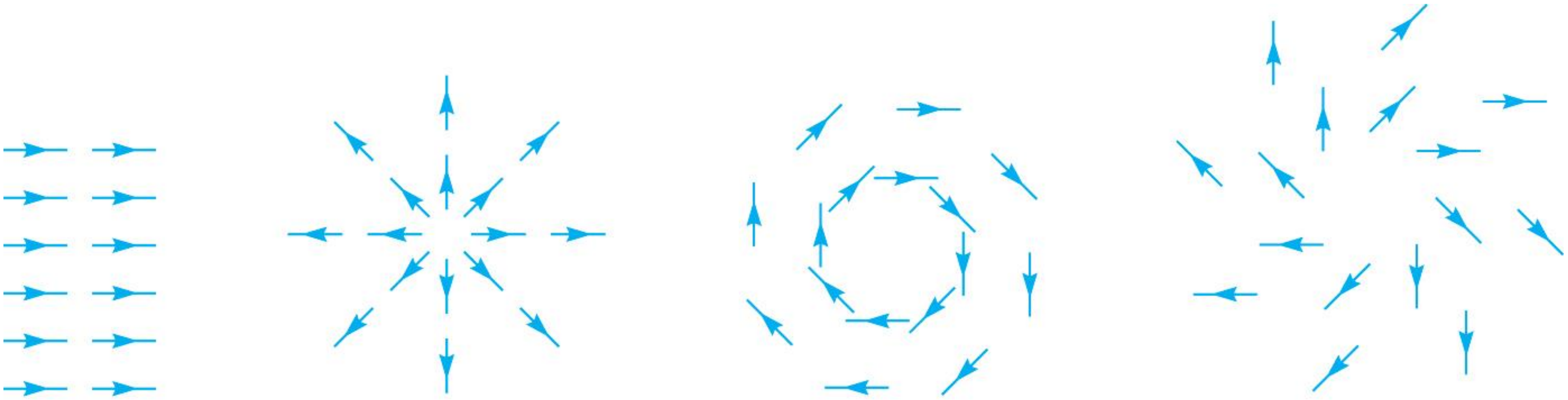
$$\sum_{i=1}^{N} \oint_{\ell_i} \mathbf{F.dl} = \sum_{i=1}^{N} (\nabla \times \mathbf{F}).\Delta\mathbf{S}_i$$

notice that the internal line integrals cancel out and only integration over the external contour remains

it follows that as $\Delta S_i \rightarrow 0$, we get

$$\oint_{\ell} \mathbf{F.dl} = \iint_{S} (\nabla \times \mathbf{F}).\mathbf{dS}$$

Classification of Vectors



(a) (b) (c) (d)

(a) $\nabla \bullet \mathbf{A} = 0, \quad \nabla \times \mathbf{A} = \mathbf{0}$

(b) $\nabla \bullet \mathbf{A} > 0, \quad \nabla \times \mathbf{A} = \mathbf{0}$

(c) $\nabla \bullet \mathbf{A} = 0, \quad \nabla \times \mathbf{A} \neq \mathbf{0}$

(d) $\nabla \bullet \mathbf{A} > 0, \quad \nabla \times \mathbf{A} \neq \mathbf{0}$

Laplacian Operator of a Scalar

the Laplacian of a scalar function is the divergence of its gradient

$$\nabla \bullet \nabla \mathrm{V} = \nabla^2 \mathrm{V} = \left(\boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z} \right) \bullet$$

$$\left(\boldsymbol{a}_x \frac{\partial \mathrm{V}}{\partial x} + \boldsymbol{a}_y \frac{\partial \mathrm{V}}{\partial y} + \boldsymbol{a}_z \frac{\partial \mathrm{V}}{\partial z} \right)$$

$$\nabla^2 \mathrm{V} = \frac{\partial^2 \mathrm{V}}{\partial x^2} + \frac{\partial^2 \mathrm{V}}{\partial y^2} + \frac{\partial^2 \mathrm{V}}{\partial z^2}$$

Laplace equation?

similar expressions can be derived through coordinate transformations for cylindrical and spherical coordinates

Important Vector Identities

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$\nabla \bullet (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times (\nabla V) = \mathbf{0}$$

in Cartesian coordinates

$$\nabla^2 \mathbf{E} = \nabla^2 E_x \mathbf{a}_x + \nabla^2 E_y \mathbf{a}_y + \nabla^2 E_z \mathbf{a}_z$$

in general

$$\nabla^2 \mathbf{E} = \nabla(\nabla \bullet \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E})$$