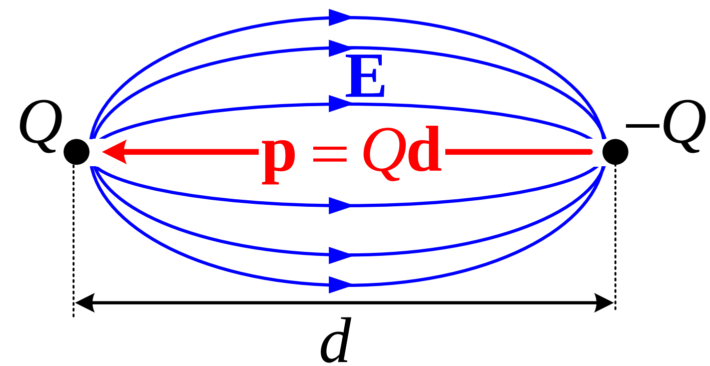
Lecture 11: Electrostatics

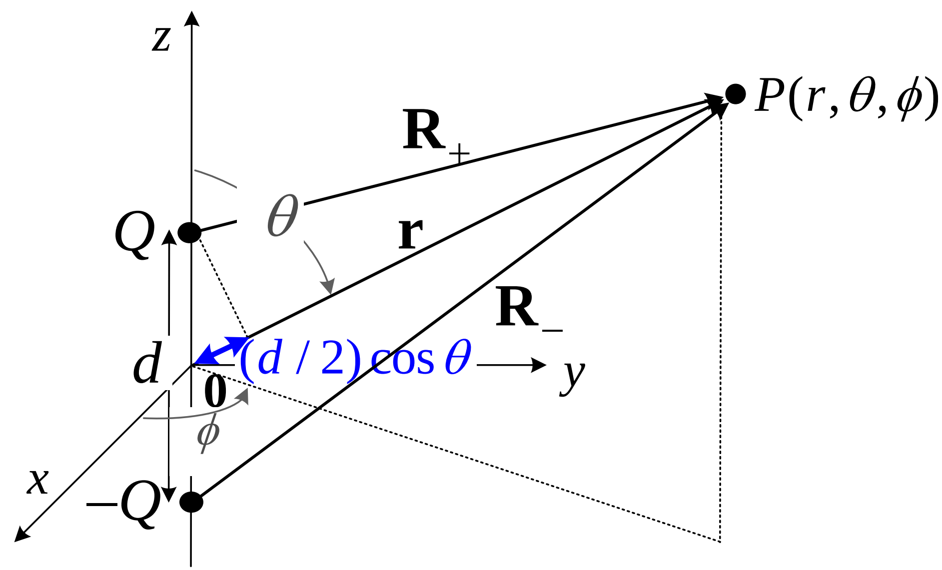
Electric dipole, electric energy, Chapter 4, pages 146-160

Electric Dipole

electric dipole: two point charges of equal charge but opposite polarity in close proximity



the dipole moment **p** is the main characteristic (from –*Q* to +*Q*!)



the dipole assumption: distance from dipole to observation point >> *d*

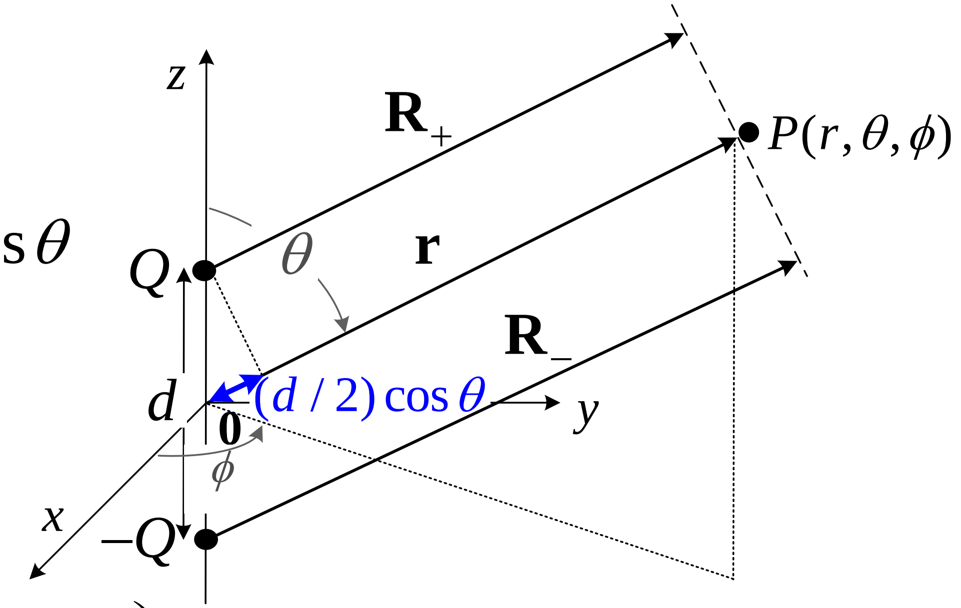
Dipole (Cont'd)

the far-field assumption leads to

$$R_+ \simeq r - \frac{d}{2}\cos\theta,\ R_- \simeq r + \frac{d}{2}\cos\theta$$

using superposition

$$V(P) = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$



$$V(P) = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right) = \frac{Q}{4\pi\varepsilon}\frac{l\cos\theta}{\left[r^2 - \left(\frac{d}{2}\cos\theta\right)^2\right]}$$

≈ 0

$$V(P) \simeq \frac{Qd\cos\theta}{4\pi\varepsilon r^2} = \frac{\mathbf{p}\cdot\mathbf{a}_r}{4\pi\varepsilon r^2}$$

Dipole (Cont'd)

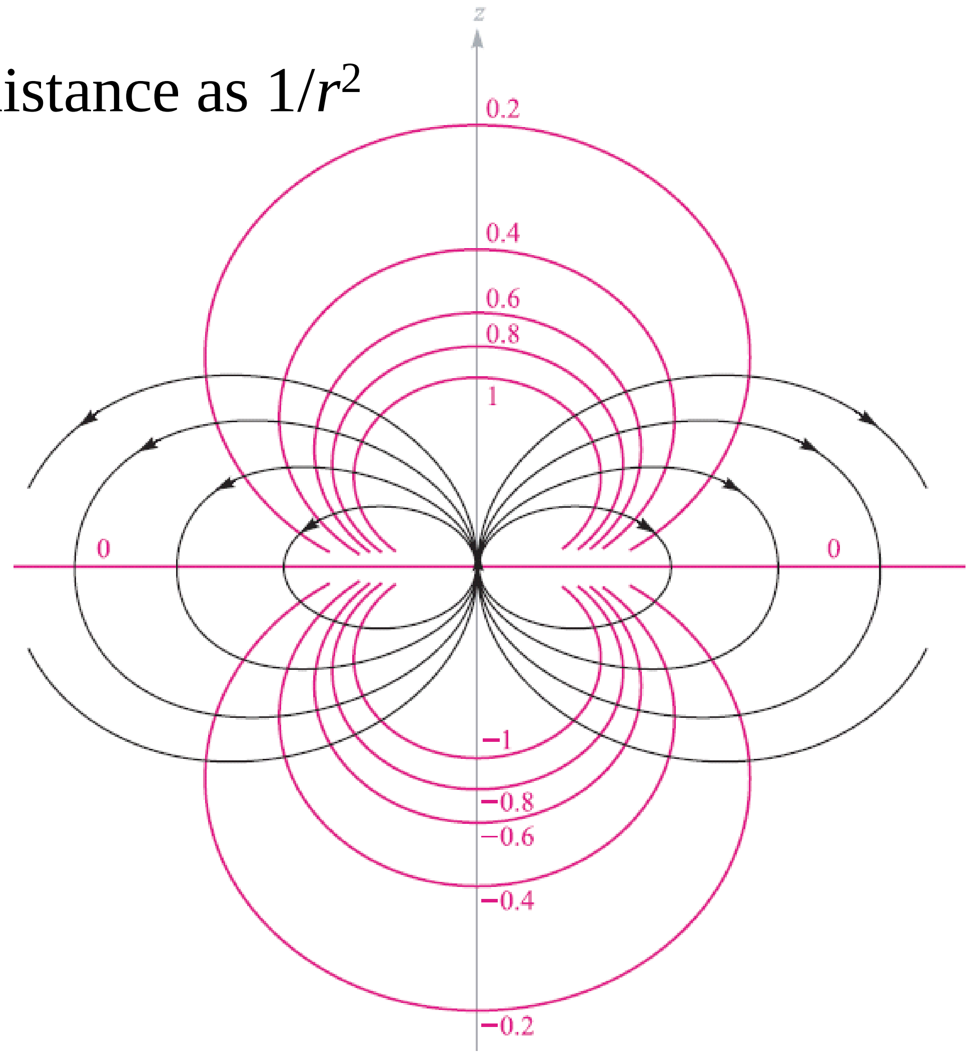
dipole potential decreases with distance as $1/r^2$

dipole potential is zero in the azimuthal plane $\theta = 90$ deg.

E field is obtained from V

$$\mathbf{E} = -\nabla V = \frac{Qd}{4\pi\varepsilon r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

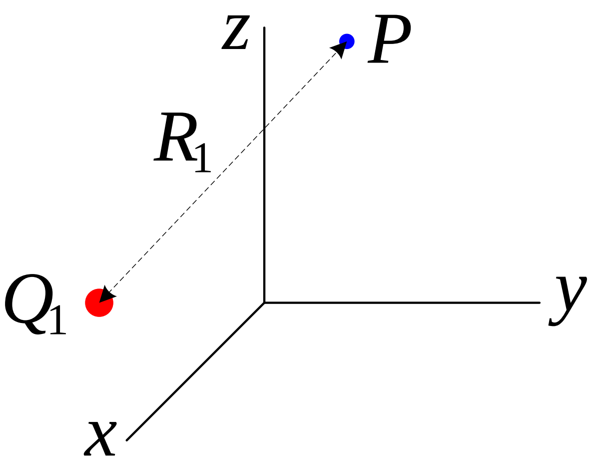
E field of dipole decreases with distance as $1/r^3$



Potential Energy

the potential energy is the work done to build the system, i.e., to bring the charges together against the Coulomb forces

if the charges of the system were set free, this energy would set them into motion, i.e., be converted into kinetic energy



the potential of at a point P due to charge Q_1

$$V_1(P) = \frac{Q_1}{4\pi\varepsilon R_1} = \int_{R_1}^{\infty} \mathbf{E}\cdot d\mathbf{L} = -\int_{\infty}^{R_1} \mathbf{E}\cdot d\mathbf{L}, \quad \text{V}$$

is the work done to bring a unit positive charge from infinity to the point P which is R_1 (meters) away from Q_1

Potential Energy (Cont'd)

if a charge Q_2 is brought at a distance R_{21} from the first charge Q_1, the energy spent is

$$W_{21} = Q_2 \cdot \frac{Q_1}{4\pi\varepsilon R_{21}} = Q_2 V_{21}, \quad \text{J=C×V}$$

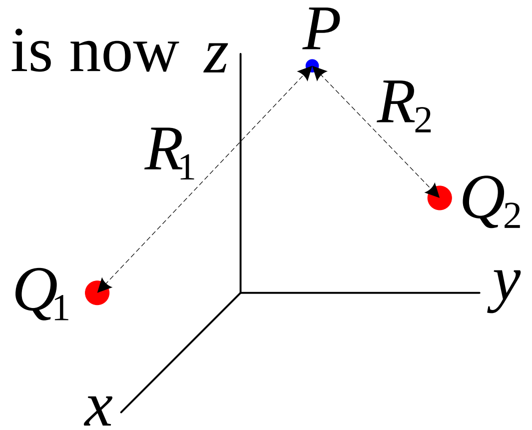
it does not matter which charge is brought close to which

$$W_{21} = W_{12}$$

as long as Q_1 and Q_2 are held apart at a distance R_{12} the system holds W_{21}=W_{12} energy

Potential Energy (Cont'd)

the potential of the (Q_1, Q_2) system at a point P is now



$$V_1(P)+V_2(P)=\frac{Q_1}{4\pi\varepsilon R_1}+\frac{Q_2}{4\pi\varepsilon R_2}$$

the work to bring over a third charge Q_3 is

$$W_{31}+W_{32}=Q_3\left(\frac{Q_1}{4\pi\varepsilon R_{31}}+\frac{Q_2}{4\pi\varepsilon R_{32}}\right)=Q_3(V_{31}+V_{32})=W_{13}+W_{23}$$

the total energy spent to built the system of three charges is

$$W_{21}+W_{31}+W_{32}=\frac{1}{4\pi\varepsilon}\left(\frac{Q_2Q_1}{R_{21}}+\frac{Q_3Q_1}{R_{31}}+\frac{Q_3Q_2}{R_{32}}\right)=$$

$$Q_2V_{21}+Q_3V_{31}+Q_3V_{32}=W_{21}+W_{31}+W_{32}=W_E$$

Potential Energy (Cont'd)

if charges are assembled in the opposite order (Q_3, then Q_2, then Q_1), we get

$$W_{23}+W_{13}+W_{12}=\frac{1}{4\pi\varepsilon}\left(\frac{Q_2Q_3}{R_{23}}+\frac{Q_1Q_3}{R_{13}}+\frac{Q_1Q_2}{R_{12}}\right)=$$

$$Q_2V_{23}+Q_1V_{13}+Q_1V_{12}=W_{23}+W_{13}+W_{12}=W_E$$

sum both results to get:

$$Q_1(V_{12}+V_{13})+Q_2(V_{21}+V_{23})+Q_3(V_{31}+V_{32})=2W_E$$

$$\Rightarrow W_E=\frac{1}{2}\sum_{i=1}^{3}Q_iV_i$$

in general, for N charges, $W_E=\frac{1}{2}\sum_{n=1}^{N}Q_nV_n$ where $V_n=\sum_{\substack{k=1\\k\neq n}}^{N}V_{n,k}$

Energy of a System

$$W_E = \frac{1}{2}\sum_{n=1}^{N} Q_n V_n \text{ where } V_n = \sum_{\substack{k=1\\k\neq n}}^{N} V_{n,k}$$

The total electrostatic energy W_E of a system of discrete charges is the sum of the energies of all possible pairs of charges.

if charge is distributed in a volume

$$W_E = \frac{1}{2}\iiint_v \rho_v V dv$$

Energy Density

$$W_E = \frac{1}{2}\iiint_v V\underbrace{(\nabla\cdot\mathbf{D})}_{\rho_v}dv = \frac{1}{2}\iiint_v [\nabla\cdot(V\mathbf{D}) - \mathbf{D}\cdot\nabla V]dv$$

$$\Rightarrow W_E = \frac{1}{2}\oiint_{S_{[v]}}(V\mathbf{D})\cdot d\mathbf{s} - \frac{1}{2}\iiint_v (\mathbf{D}\cdot\underbrace{\nabla V}_{-\mathbf{E}})dv$$

to account for all energy allow the volume *v* to expand to infinity

$$|\mathbf{D}| \sim 1/R^2,\ \ V \sim 1/R,\ \ |d\mathbf{s}| \sim R^2 \Rightarrow \lim_{R\to\infty}\oiint_{S_{[v]}}(V\mathbf{D})\cdot d\mathbf{s} = 0$$

$$\Rightarrow \quad W_E = \frac{1}{2}\iiint_{v\to\infty}(\mathbf{D}\cdot\mathbf{E})dv, \ \ \text{J}$$

Energy Density (Cont'd)

energy integral in an isotropic medium where $\mathbf{D} = \varepsilon\mathbf{E}$

$$W_E = \frac{1}{2}\iiint_{v\to\infty} \varepsilon\,|\mathbf{E}|^2\,dv \qquad W_E = \frac{1}{2}\iiint_{v\to\infty} \frac{|\mathbf{D}|^2}{\varepsilon}\,dv$$

energy density and the field vectors

$$w_E = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}, \quad \text{J/m}^3$$

energy density in an isotropic medium

$$w_E = \frac{1}{2}\varepsilon\,|\mathbf{E}|^2 \qquad w_E = \frac{1}{2}\frac{|\mathbf{D}|^2}{\varepsilon}$$