Lecture 17: Electrostatics

Method of Images, Applications, Chapter 6, pages 256-267

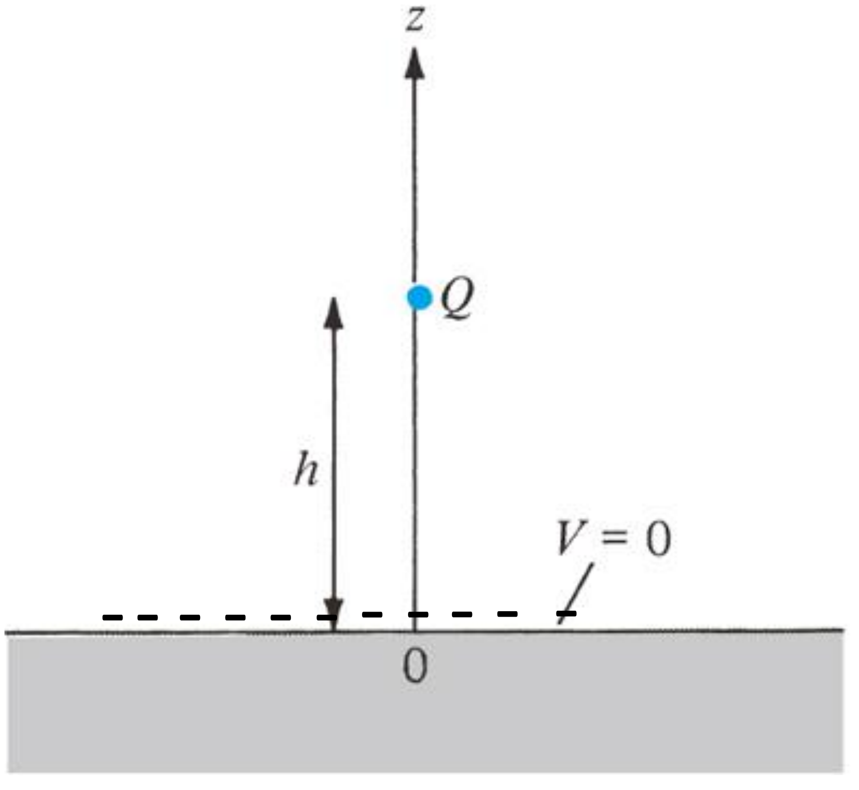
Uniqueness Theorem

if $V_1(\boldsymbol{r})$ is a solution for Laplace equation that satisfies the boundary conditions and if $V_2(\boldsymbol{r})$ is also a solution for Laplace equation that satisfies the boundary conditions, then we must have $V_1(\boldsymbol{r}) = V_2(\boldsymbol{r})$ (solution is unique)

keeping the same boundary conditions and the same charge distribution results in the same solution

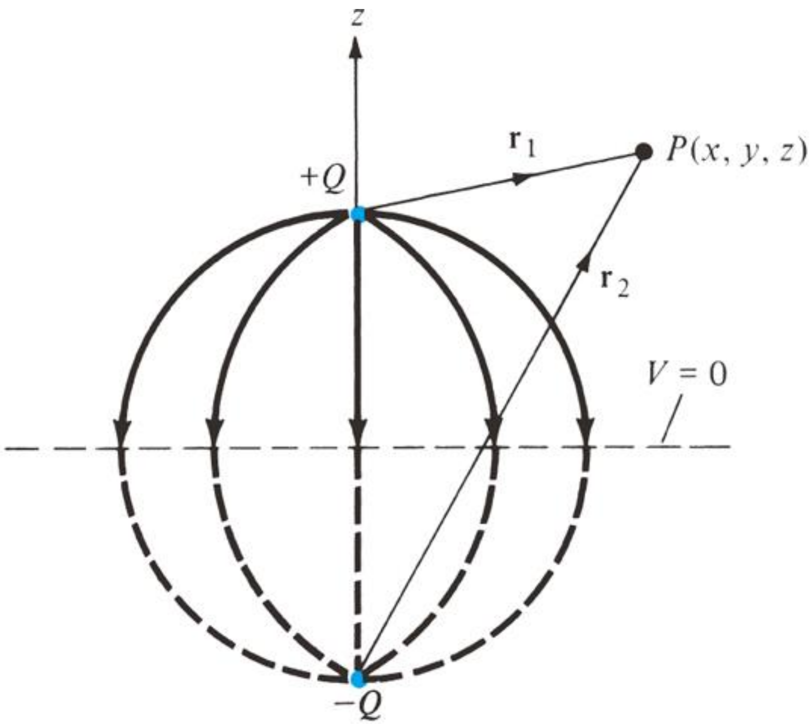
simpler problems can be solved through using image charges

A Point Charge above a Ground Plane



the positive charge induces negative surface charges resulting in a difficult problem to solve as ρ_s is unknown!

The Equivalent Problem



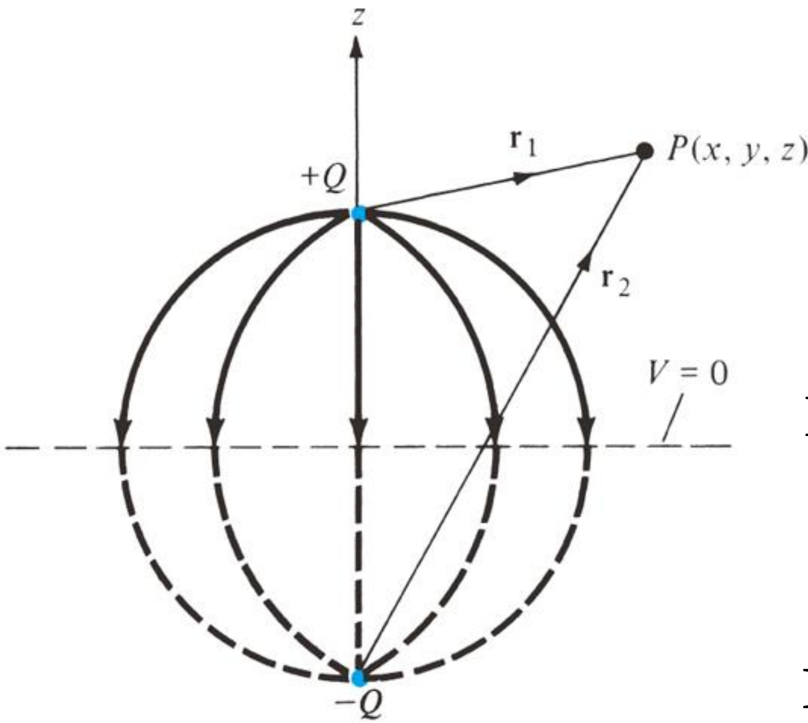
adding a negative charge $-Q$ satisfies the boundary conditions for the top half space

the charge distribution is unchanged in the top half space

it follows, by the Uniqueness Theorem, that the solution will be exactly the same for $z \geq 0$!

the field anywhere in the top half space is the sum of the fields resulting from the charge and its image!

The Equivalent Problem (Cont'd)



positive charge $+Q$ at $(0,0, h)$ and negative charge $-Q$ at $(0,0, -h)$

$\mathbf{E}=\mathbf{E}^{+}+\mathbf{E}^{-}$ (for $z\geq 0$)

$$\mathbf{E}^{+}=\frac{Q}{4\pi\varepsilon_{o}}\left[\frac{x\boldsymbol{a}_{x}+y\boldsymbol{a}_{y}+(z-h)\boldsymbol{a}_{z}}{\left(x^{2}+y^{2}+(z-h)^{2}\right)^{3/2}}\right]$$

$$\mathbf{E}^{-}=\frac{-Q}{4\pi\varepsilon_{o}}\left[\frac{x\boldsymbol{a}_{x}+y\boldsymbol{a}_{y}+(z+h)\boldsymbol{a}_{z}}{\left(x^{2}+y^{2}+(z+h)^{2}\right)^{3/2}}\right]$$

at z=0, we have $$\mathbf{E}=\frac{-2hQ}{4\pi\varepsilon_{o}}\left[\frac{\boldsymbol{a}_{z}}{\left(x^{2}+y^{2}+h^{2}\right)^{3/2}}\right]$$

The Equivalent Problem (Cont'd)

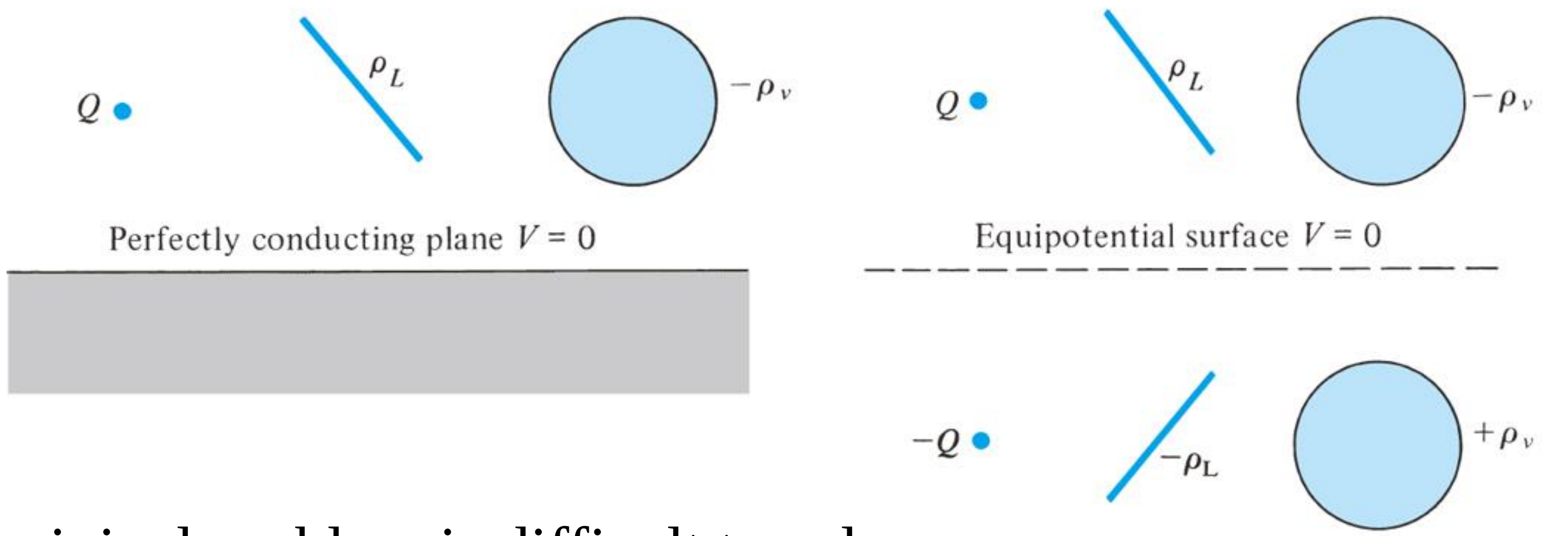
at z=0, we have

$$\mathbf{D} = \frac{-hQ}{2\pi}\left[\frac{\boldsymbol{a}_z}{\left(x^2+y^2+h^2\right)^{3/2}}\right] \Rightarrow \rho_s(x,y) = D_n = \left[\frac{-hQ}{2\pi\left(x^2+y^2+h^2\right)^{3/2}}\right]$$

integrating this surface charge density over the whole surface of the ground plane we get –Q!

the flux diverging out from the positive charge Q terminates at the distributed negative surface charge –Q!

The General Case



original problem is difficult to solve

add image charges (discrete, linear, or volumetric) to maintain the same boundary conditions and solve a much simpler problem

solution is valid only for $z \geq 0$!