Lecture 20: Magnetostatics

Magnetic flux density, magnetic Scalar and vector potential, applications, Chapter 7, pages 301-310, 312-316

Magnetic Flux Density B

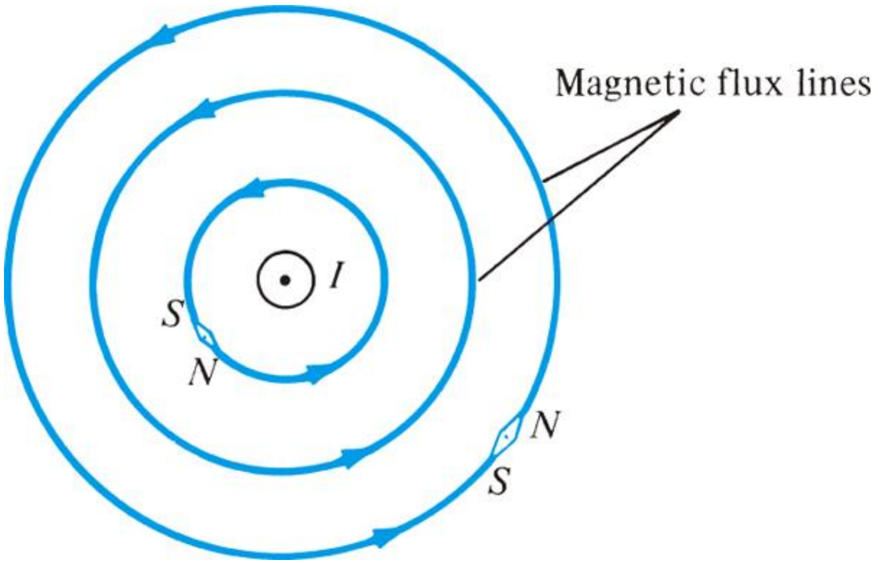
similar to the electrostatics case, the magnetic flux density **B** is a function of both magnetic field intensity and the material

$\mathbf{B} = \mu \mathbf{H}$ Wb/m^2

where μ is the permeability of the medium

for free space $\mu = \mu_o = 4\pi \times 10^{-7}$ H/m

magnetic flux lines are lines in the direction of the field **B**



Magnetic Flux

the magnetic flux through a surface is given by

$$\psi = \iint_S \mathbf{B}.\mathrm{d}\mathbf{s} \quad \text{Wb}$$

because magnetic field lines are closed lines, we have

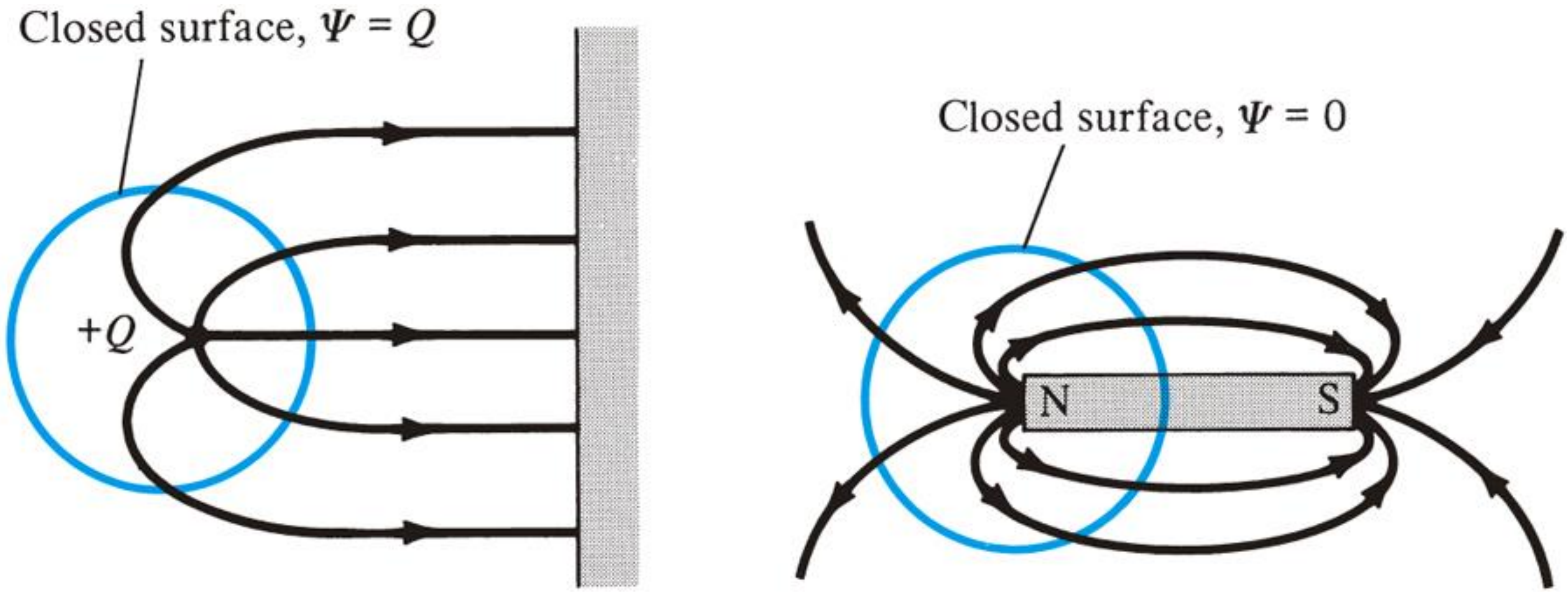
$$\psi = \oiint \mathbf{B}.\mathrm{d}\,\mathbf{s} = 0$$

applying divergence theorem, we get

$$\psi = \oiint \mathbf{B}.\mathrm{d}\,\mathbf{s} = \iiint_V (\nabla.\mathbf{B})dv = 0$$

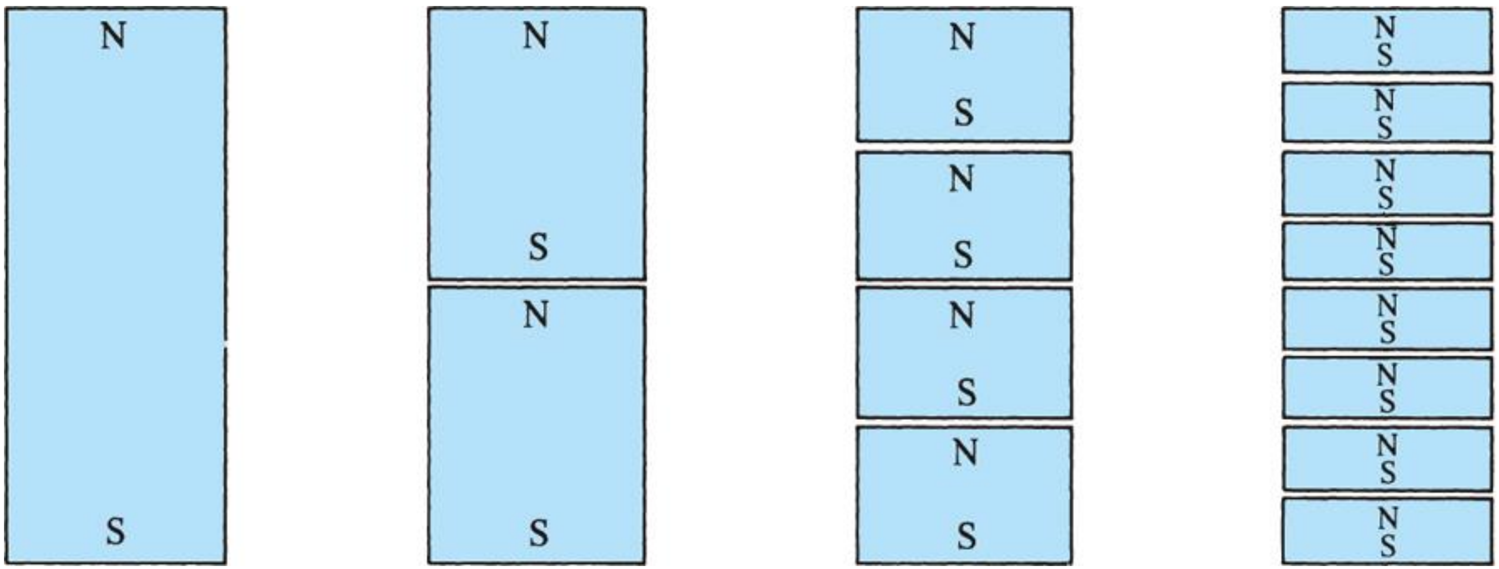
$\Rightarrow \quad \nabla.\mathbf{B} = 0$ magnetic field has a zero divergence!

Magnetic Flux (Cont'd)



magnetic charges have not been discovered!

Magnetic Flux (Cont'd)



single magnetic poles have not been discovered either!

Maxwell's Equations for Static Fields

Integral form	Differential form
$\oiint_S \boldsymbol{D}.d\boldsymbol{S} = \iiint_V \rho_v \, dV = Q$	$\nabla.\boldsymbol{D} = \rho_v$
$\oiint_S \boldsymbol{B}.d\boldsymbol{S} = 0$	$\nabla.\boldsymbol{B} = 0$
$\oint_C \boldsymbol{E}.d\boldsymbol{l} = 0$	$(\nabla \times \boldsymbol{E}) = \boldsymbol{0}$
$\oint_C \boldsymbol{H}.d\boldsymbol{l} = \iint_S \boldsymbol{J}.d\boldsymbol{S}$	$(\nabla \times \boldsymbol{H}) = \boldsymbol{J}$

Magnetic Scalar Potential

In regions with no current, we have $(\nabla \times \boldsymbol{H}) = \mathbf{0}$

in this case, we can define the magnetic scalar potential

$$\mathbf{H} = -\nabla V_m$$

just like the scalar electric potential, the magnetic scalar potential satisfies Laplace equation

$$\nabla^2 V_m = 0, \quad (\mathbf{J} = \mathbf{0})$$

this scalar potential is not widely used in driving magnetic field solutions as the magnetic vector potential

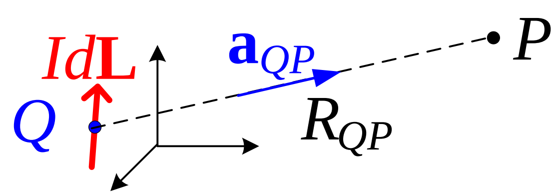
Magnetic Vector Potential

the magnetic flux density **B** can be represented as the curl the magnetic vector potential **A**

$$\mathbf{B} = \nabla \times \mathbf{A}$$

this can be proven using the Biot-Savart law

$$\mathbf{B}(P) = \oint_{C_Q} \frac{\mu}{4\pi} I_{(Q)} d\mathbf{L}_Q \times \frac{\mathbf{a}_{QP}}{R_{QP}^2},$$



we can show that $$\frac{\mathbf{a}_{QP}}{R_{QP}^2} = -\nabla\left(\frac{1}{R_{QP}}\right)$$

Magnetic Vector Potential (Cont'd)

$$\mathbf{R}_{QP} = (x_P - x_Q)\mathbf{a}_x + (y_P - y_Q)\mathbf{a}_y + (z_P - z_Q)\mathbf{a}_z$$

$$R_{QP} = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2}$$

$$\nabla\left(\frac{1}{R_{QP}}\right) = -\frac{1}{R_{QP}^2}\left(\frac{\partial R_{QP}}{\partial x_P}\mathbf{a}_x + \frac{\partial R_{QP}}{\partial y_P}\mathbf{a}_y + \frac{\partial R_{QP}}{\partial z_P}\mathbf{a}_z\right)$$

$$\frac{\partial R_{QP}}{\partial x_P} = \frac{1}{2}\cdot\frac{2(x_P - x_Q)}{R_{QP}} = \frac{(x_P - x_Q)}{R_{QP}}$$

$$\Rightarrow \nabla\left(\frac{1}{R_{QP}}\right) = -\frac{\mathbf{a}_{QP}}{R_{QP}^2}$$

$$\mathbf{a}_{QP} = \frac{(x_P - x_Q)\mathbf{a}_x + (y_P - y_Q)\mathbf{a}_y + (z_P - z_Q)\mathbf{a}_z}{R_{QP}}$$

Magnetic Vector Potential (Cont'd)

$$\Rightarrow \mathbf{B}(P) = -\oint_C \frac{\mu}{4\pi} I_{(Q)} d\mathbf{L}_Q \times \nabla\left(\frac{1}{R_{QP}}\right)$$

using the vector identity $\mathbf{F}\times\nabla V = V\nabla\times\mathbf{F} - \nabla\times(V\mathbf{F})$

$$\Rightarrow \mathbf{B}(P) = -\oint_C \left[\frac{1}{R_{QP}}\ \underbrace{\nabla\times\left(\frac{\mu}{4\pi} I_{(Q)} d\mathbf{L}_Q\right)}_{\mathbf{0}} - \ \nabla\times\left(\frac{\mu}{4\pi}\frac{I_{(Q)} d\mathbf{L}_Q}{R_{QP}}\right)\right]$$

$$\Rightarrow \mathbf{B}(P) = \oint_C \nabla\times\left(\frac{\mu}{4\pi}\frac{I_{(Q)} d\mathbf{L}_Q}{R_{QP}}\right)$$

curl operator does not depend on the point of integration

$$\mathbf{B}(P) = \nabla\times\underbrace{\oint_C \left(\frac{\mu}{4\pi}\frac{I_{(Q)} d\mathbf{L}_Q}{R_{QP}}\right)}_{\mathbf{A}(P)}$$

Magnetic Vector Potential (Cont'd)

magnetic vector potential due to line currents

$$\mathbf{A} = \frac{\mu}{4\pi}\oint_C \frac{Id\mathbf{L}}{R}, \text{ Wb/m}$$

analogous expressions are obtained for surface and volume current distributions

$$\mathbf{A} = \frac{\mu}{4\pi}\iint_S \frac{\mathbf{K}}{R}ds, \qquad \mathbf{A} = \frac{\mu}{4\pi}\iiint_v \frac{\mathbf{J}}{R}dv$$

analogy with electric scalar potential $V = \dfrac{1}{4\pi\varepsilon}\iiint_v \dfrac{\rho_v}{R}dv$

Magnetic Flux

flux through a closed surface

$$\psi = \oiint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

flux through open surface and **A**

$$\psi = \iint_S \mathbf{B} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$$

$$\Rightarrow \psi = \oint_C \mathbf{A} \cdot d\mathbf{L}$$