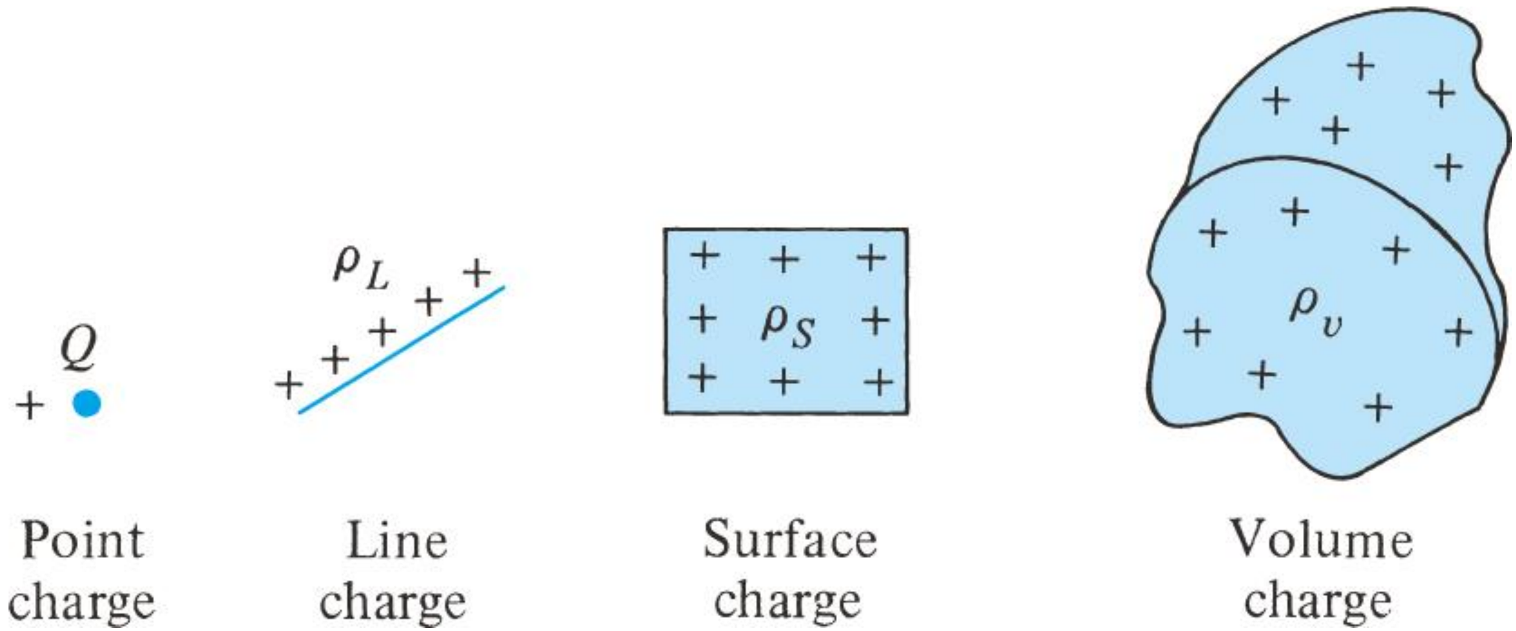
Lecture 8: Electrostatics

Superposition: line charges, surface charges, volume charges, **Chapter 4:** pages 115-126

Charge Distributions



line charge: charges are aligned in a line. linear charge density C/m

surface charge: charges are distributed over a surface. surface charge density C/m^2

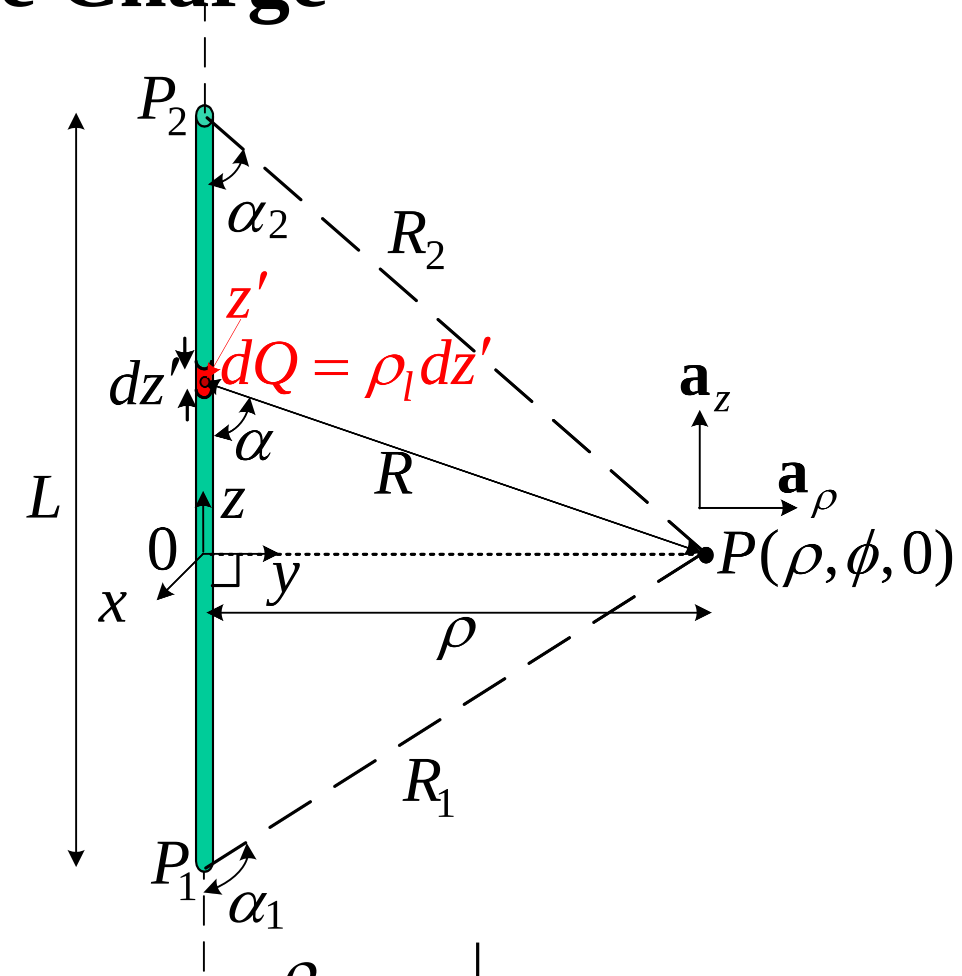
volume charge: charges distributed over a volume. volumetric charge density C/m^3

ESF of a Uniform Line Charge

the line charge is represented by an infinite number of infinitesimal line elements of charge

$$dQ = \rho_l dz'$$

the field of the line charge is a superposition of the fields created by each linear charge element *dQ*.



$$R = \frac{\rho}{\sin\alpha}, \quad z' = \rho\cot\alpha \Rightarrow dz' = -\frac{\rho}{\sin^2\alpha}d\alpha$$

ESF of a Uniform Line Charge (Cont'd)

the contribution of each differential element is

$$d\mathbf{E} = \frac{\rho_l dz'}{4\pi\varepsilon} \cdot \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

$$\mathbf{r}' = z'\mathbf{a}_z,\ \rho'=0$$

$$\mathbf{r} = \rho\mathbf{a}_\rho,\ z = 0,\ 0 \le \phi < 2\pi$$

problem is cylindrically symmetric – choice of ϕ does not matter. assume ϕ = 0 and work in the ρ-z plane.

$$\mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

$$\Rightarrow \quad dE_\rho = \frac{\rho_l dz'}{4\pi\varepsilon} \cdot \frac{\rho}{R^3}, \quad dE_z = -\frac{\rho_l dz'}{4\pi\varepsilon} \cdot \frac{z'}{R^3}$$

ESF of a Uniform Line Charge (Cont'd)

integrating along z', the directions of $\mathbf{a}_\rho$ and $\mathbf{a}_z$ do not change (why?)

$$\mathbf{E} = \mathbf{a}_\rho \underbrace{\int_{z'=z_{\min}}^{z_{\max}} \left(\frac{\rho_l}{4\pi\varepsilon} \cdot \frac{\rho}{R^3} \right) dz'}_{dE_\rho} + \mathbf{a}_z \underbrace{\int_{z'=z_{\min}}^{z_{\max}} \left(-\frac{\rho_l}{4\pi\varepsilon} \cdot \frac{z'}{R^3} \right) dz'}_{dE_z}$$

changing variable to α, we have

$$\Rightarrow \mathbf{E} = \frac{\rho_l}{4\pi\varepsilon} \cdot \frac{1}{\rho} \cdot \left[\mathbf{a}_\rho \int_{\alpha_1}^{\alpha_2} (-\sin\alpha) d\alpha + \mathbf{a}_z \int_{\alpha_1}^{\alpha_2} \cos\alpha \, d\alpha \right]$$

ESF of a Uniform Line Charge (Cont'd)

$$E_\rho = \frac{\rho_l}{4\pi\varepsilon} \cdot \frac{1}{\rho} \cdot \left(\cos\alpha_2 - \cos\alpha_1\right)$$

$$E_z = \frac{\rho_l}{4\pi\varepsilon} \cdot \frac{1}{\rho} \cdot \left(\sin\alpha_2 - \sin\alpha_1\right)$$

for an in<u>finite and uniform line charge</u>: $\alpha_1 = 180^\circ, \alpha_2 = 0^\circ$

$$\Rightarrow E_\rho = \frac{1}{2\pi\varepsilon} \cdot \frac{\rho_l}{\rho},\ E_z = 0$$

the contributions of the upper and lower halves of the infinite line charge add up in the radial direction while canceling in the *z* direction.

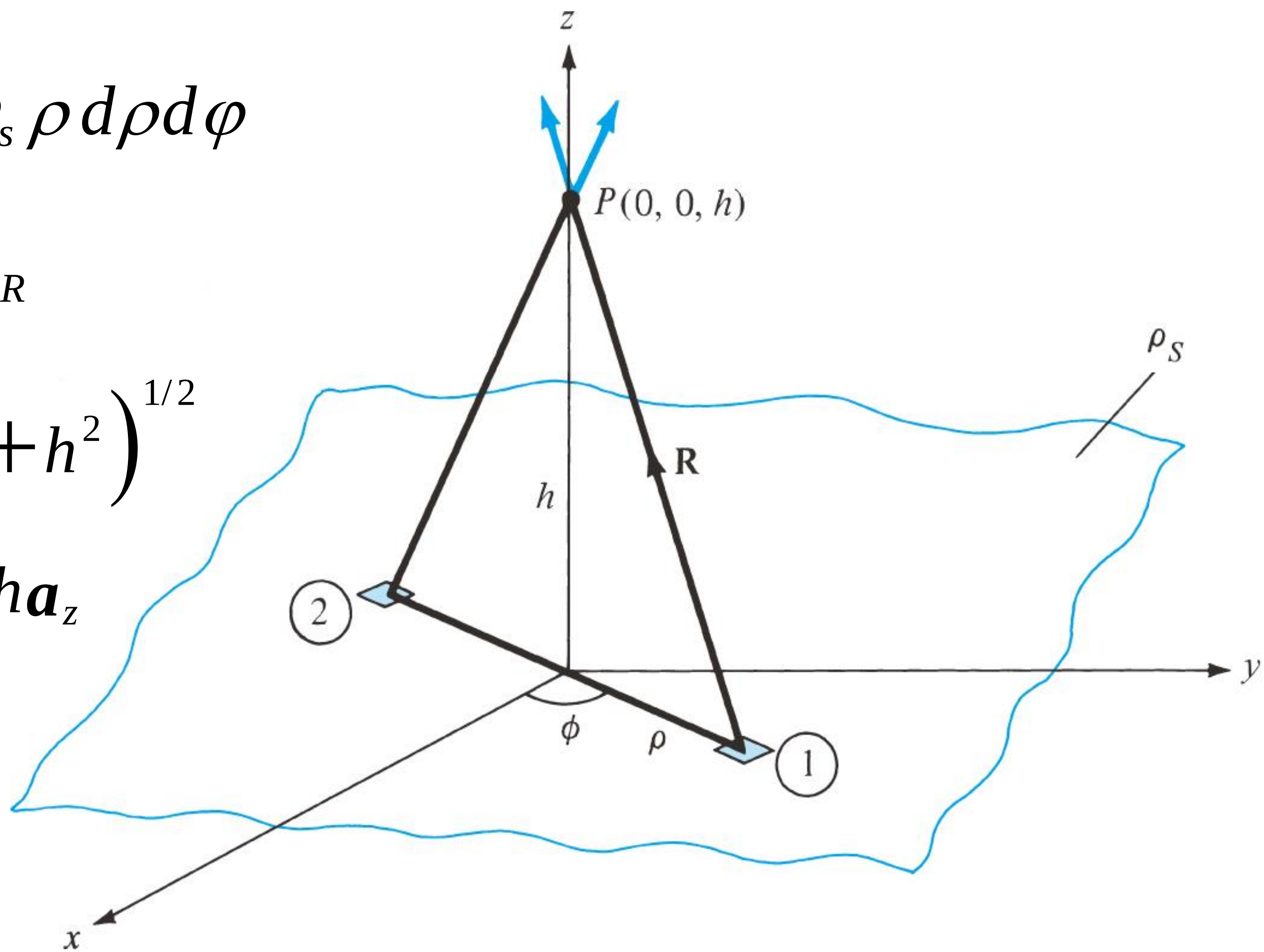
Infinite Plane with a Uniform Surface Charge

$$dQ = \rho_s\, ds = \rho_s\, \rho\, d\rho\, d\varphi$$

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon}\frac{dQ}{R^2}\mathbf{a}_R$$

$$R = |\boldsymbol{R}| = \left(\rho^2 + h^2\right)^{1/2}$$

$$\boldsymbol{R} = \rho(-\boldsymbol{a}_\rho) + h\boldsymbol{a}_z$$



Infinite Plane (Cont'd)

$$d\mathbf{E} = \frac{\rho_s \rho\, d\rho\, d\varphi(\rho(-\mathbf{a}_\rho) + h\mathbf{a}_z)}{4\pi\varepsilon_o\left(\rho^2 + h^2\right)^{3/2}}$$

because of symmetry, ρ component of total electric field vanishes!

$$\mathbf{E} = \int_0^{2\pi}\int_0^{\infty} \frac{\rho_s h \rho\, d\rho\, d\varphi}{4\pi\varepsilon_o\left(\rho^2 + h^2\right)^{3/2}}\mathbf{a}_z = \frac{\rho_s}{4\pi\varepsilon_o}\int_0^{2\pi}\int_0^{\infty} \frac{h\rho\, d\rho\, d\varphi}{\left(\rho^2 + h^2\right)^{3/2}}\mathbf{a}_z$$

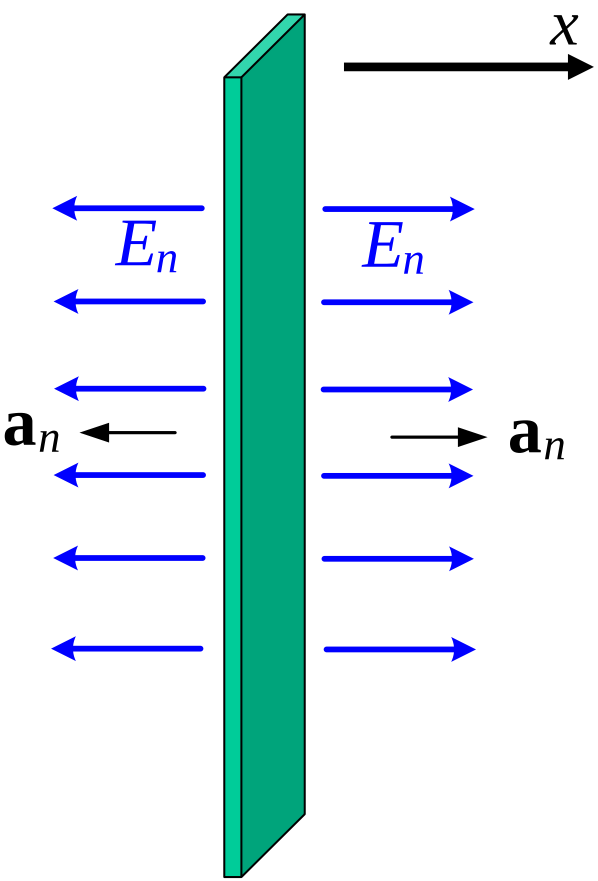
$$\mathbf{E} = \frac{\rho_s h}{2\varepsilon_o}\int_0^{\infty} \frac{\rho\, d\rho}{\left(\rho^2 + h^2\right)^{3/2}}\mathbf{a}_z = \frac{\rho_s h}{2\varepsilon_o}\left\{-\left(\rho^2 + h^2\right)^{-1/2}\right\}\Bigg|_0^{\infty}\mathbf{a}_z$$

$$\mathbf{E} = \frac{\rho_s}{2\varepsilon_o}\mathbf{a}_z \quad \text{independent of } h!$$

Infinite Plane (Cont'd)

$$E_x = \begin{cases} \dfrac{\rho_s}{2\varepsilon}, & x \geq 0 \\ -\dfrac{\rho_s}{2\varepsilon}, & x \leq 0 \end{cases}$$

for a plane with finite area, the field is not uniform and is a function of position



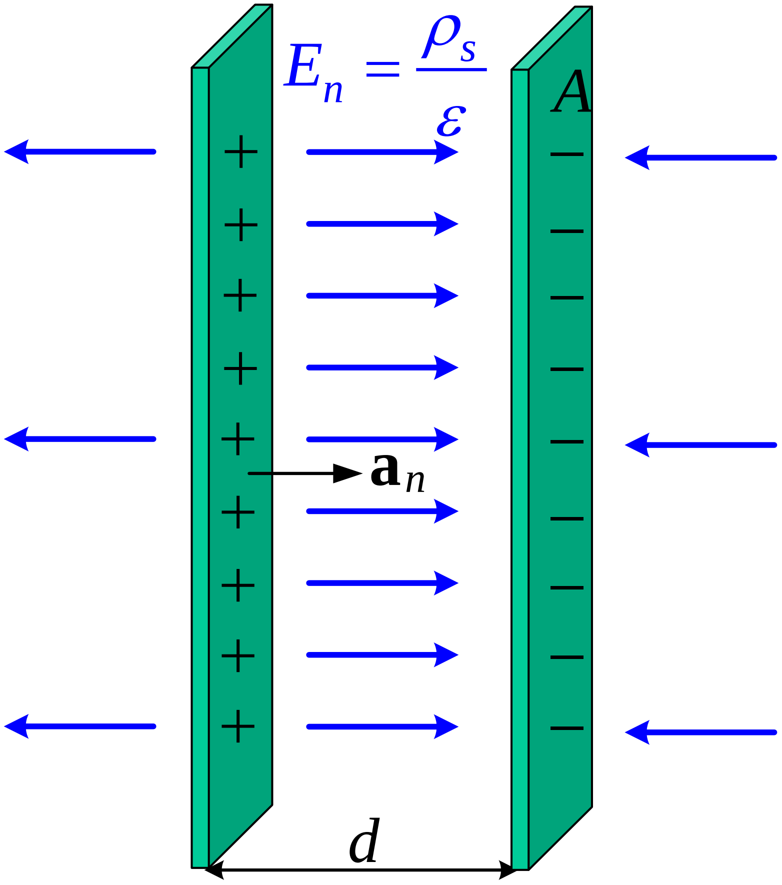
Parallel Plate Capacitor

two infinite planes with opposite uniform surface distributions

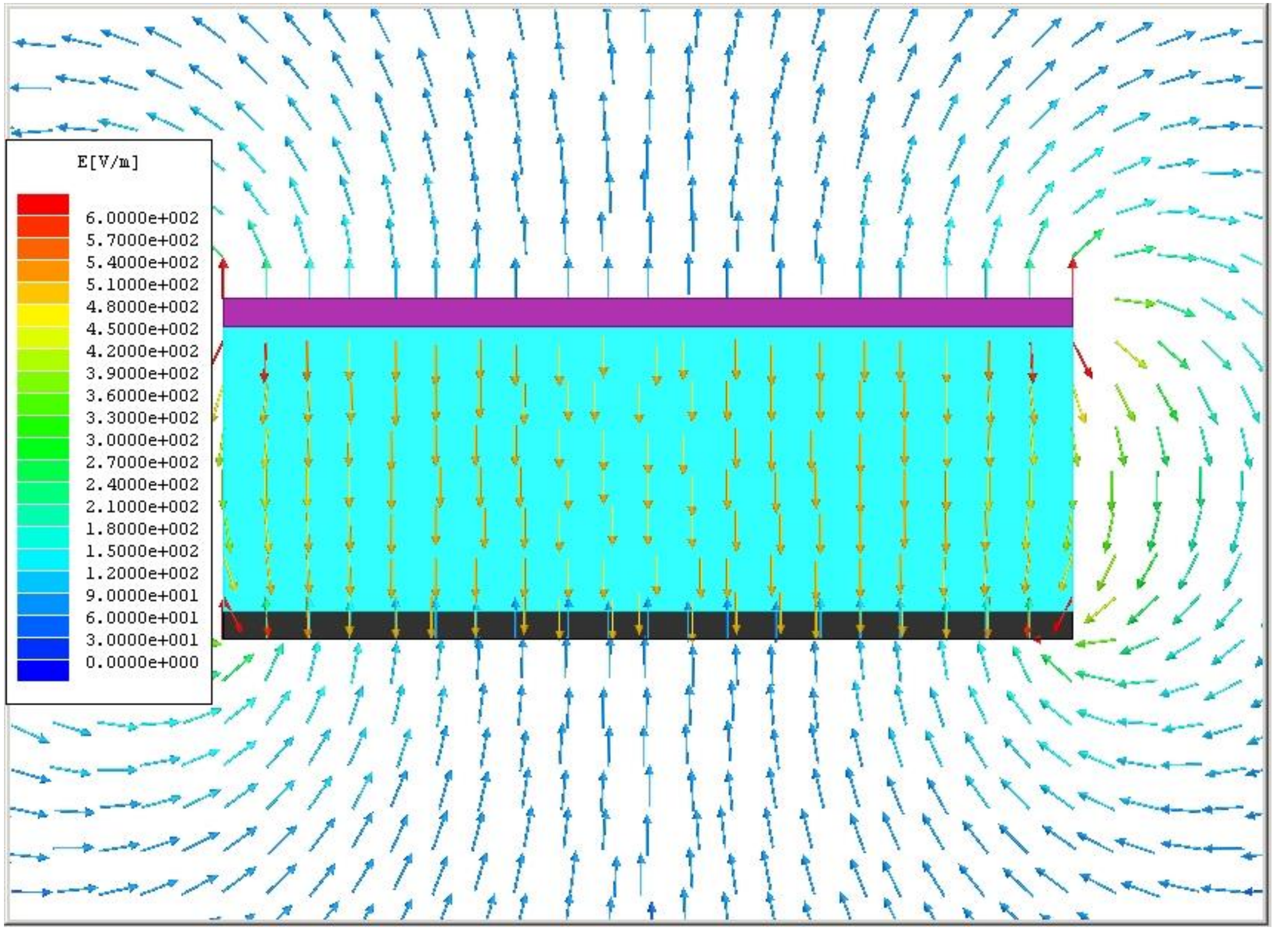
field between planes double while field outside vanishes!

$$\mathbf{E}_n = \frac{\rho_s}{\varepsilon}\mathbf{a}_n$$

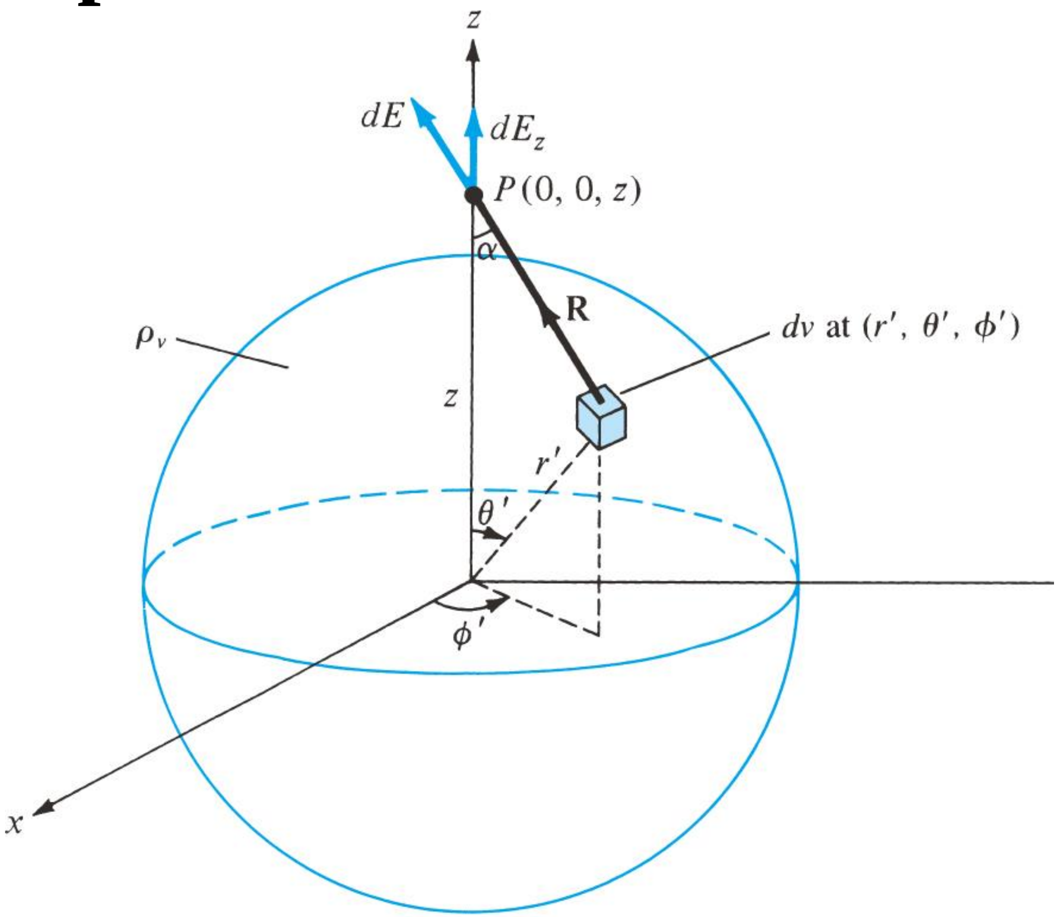
in practice, the area of the plates are finite and there is fringing fields!



Field Plot of Capacitor



Sphere with Uniform Volumetric Density



$$d\mathbf{E} = \frac{1}{4\pi\varepsilon}\frac{dQ}{R^2}\mathbf{a}_R$$

$$dQ = \rho_v dv$$

$$dv = r'^2 \sin\theta' dr' d\theta' d\varphi'$$

$$\boldsymbol{a}_R = \cos\alpha\, \boldsymbol{a}_z + \sin\alpha\, \boldsymbol{a}_\rho$$

because of symmetry, ρ component vanishes!

$$\mathbf{E}_z = \frac{\rho_v}{4\pi\varepsilon_o}\iiint \frac{\cos\alpha \; dv}{R^2}\mathbf{a}_z$$

for a certain φ, the field contribution is defined by r' and R

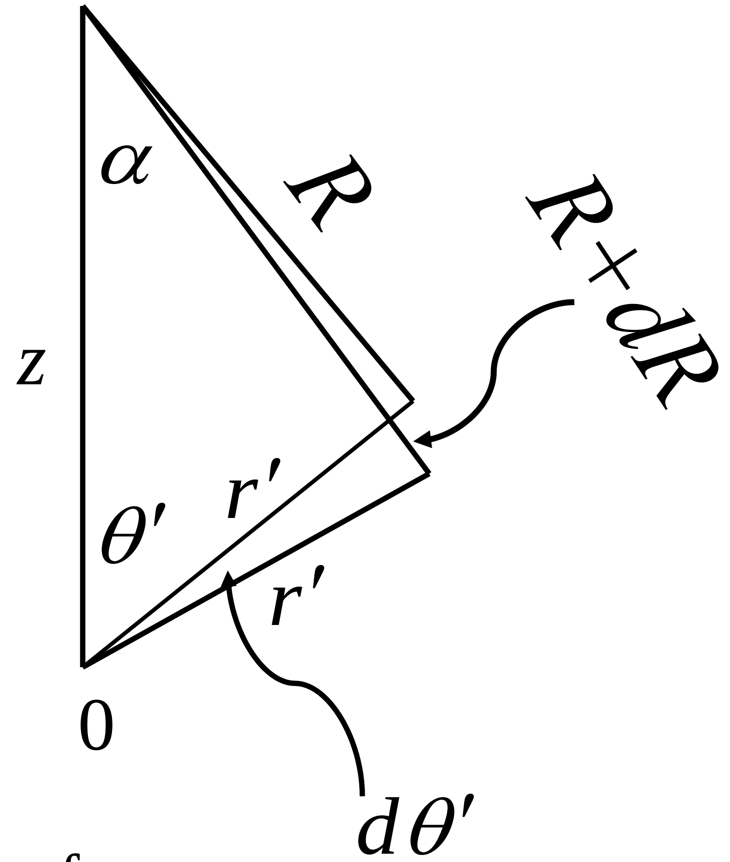
Sphere (Cont'd)

using cosine rule for triangles

$$\cos\alpha = \frac{z^2 + R^2 - r'^2}{2zR}$$

$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'}$$

$$\sin\theta' d\theta' = \frac{RdR}{zr'}$$



as θ' changes from 0 to π, R changes from

$z - r'$ to $z + r'$ (why is z greater than r'?)

Sphere (Cont'd)

$$E_z = \frac{\rho_v}{4\pi\varepsilon_o} \int_{\varphi'=0}^{2\pi} d\varphi' \int_{r'=0}^{a} \int_{R=z-r'}^{R=z+r'} r'^2 \frac{RdR}{zr'} dr' \frac{z^2 + R^2 - r'^2}{2zR} \frac{1}{R^2}$$

⇓

$$E_z = \frac{\rho_v}{4\varepsilon_o z^2} \int_{r'=0}^{a} \int_{R=z-r'}^{R=z+r'} r' \frac{z^2 + R^2 - r'^2}{R^2} dRdr'$$

$$E_z = \frac{\rho_v}{4\varepsilon_o z^2} \int_{r'=0}^{a} \int_{R=z-r'}^{R=z+r'} r' \left(1 + \frac{z^2 - r'^2}{R^2}\right) dRdr'$$

Sphere (Cont'd)

$$E_z = \frac{\rho_v}{4\varepsilon_o z^2} \int_{r'=0}^{a} r' \left(R - \frac{z^2 - r'^2}{R} \right) \Bigg|_{R=z\text{-}r'}^{R=z+r'} dr'$$

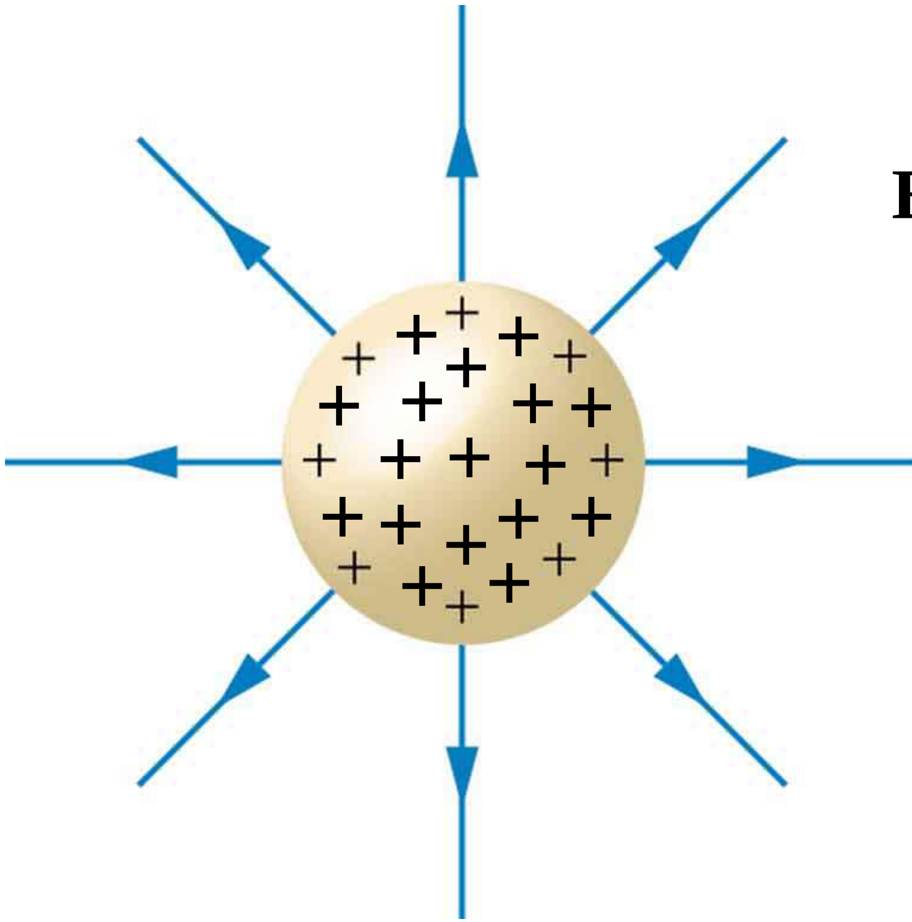
$$E_z = \frac{\rho_v}{4\varepsilon_o z^2} \int_{r'=0}^{a} 4r'^2 dr'$$

$$E_z = \frac{1}{4\pi\varepsilon_o z^2} \left(\frac{4}{3}\pi a^3 \rho_v \right) = \frac{Q}{4\pi\varepsilon_o z^2}$$

in general, for any point outside the sphere at a distance r from the centere

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2} \mathbf{a}_r$$ **equivalent charge at the origin!**

Sphere with Uniform Charge (Cont'd)



$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_r$$