Final Exam Solutions 2FH3 - 2010

(1) [9 points]

$$\vec{D} = 20\rho^2 \hat{a}_\rho \times 10^{-9} \ C/m^2$$

Method #1:

$$\nabla \cdot \vec{D} = \rho_v \ , \quad D_\rho = 20\rho^2 \times 10^{-9}$$
$$D_z = D_\phi = \emptyset$$

$$\nabla \cdot \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(20\rho^3 \, 10^{-9})$$

$$\Rightarrow \nabla \cdot \vec{D} = \frac{1}{\rho} \, 20\times 10^{-9} \times 3\rho^2 = \rho_v$$

$$\Rightarrow \rho_v = 60\rho \ 10^{-9} \ , \ C/m^3$$

$$Q = \iiint \rho_v \, dv = \int_0^2\int_0^3\int_0^{2\pi} 60\rho \, 10^{-9} \rho \, d\phi \, d\rho \, dz$$

$$\Rightarrow Q = \cancel{60}^{20} \times 10^{-9} \times 2\pi \times 2 \times \frac{\rho^3}{\cancel{3}_1}\Big|_0^3$$

$$\Rightarrow Q = 80\pi \ 10^{-9} \times 27$$

$$\Rightarrow Q \approx 6.786 \times 10^{-6} \ C$$

Method #2:

$$Q_{encl} = \oiint_S \vec{D} \cdot d\vec{s} = \int_0^2\int_0^{2\pi} 20\rho^2 10^{-9} \underbrace{\hat{a}_\rho \cdot \hat{a}_\rho}_{1} \rho \, d\phi \, dz \Big/_{\rho=3}$$

$$\Rightarrow Q_{encl} = 20 \times 10^{-9} \times 2\pi \times 2 \times 27$$

$$Q_{encl} \approx 6.786 \times 10^{-6} \ C$$

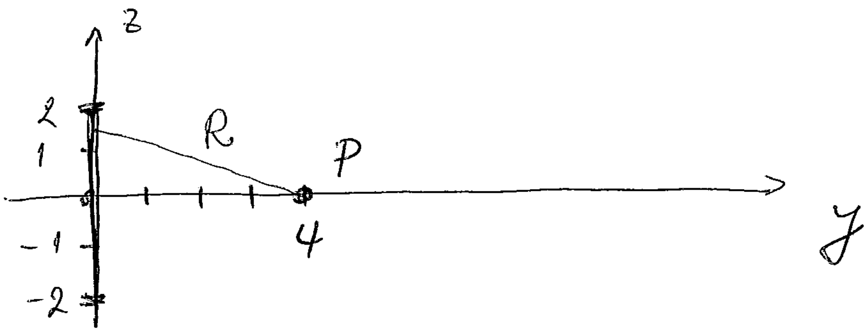
2 [42 points]

$$\rho_e = \begin{cases} |z| \times 10^{-9} & , \text{ for } |z| \le 2 \\ 0 & , \text{ for } |z| > 2 \end{cases}$$

(a) [4 points] $Q = \int \rho_e \, dl = \int_{-2}^{2} |z| \times 10^{-9} \, dz$

$$\Rightarrow Q = 2 \times 10^{-9} \int_0^2 z \, dz = 2 \times 10^{-9} \left. \frac{z^2}{2} \right|_0^2 = \underline{\underline{4 \times 10^{-9} \text{ C}}}$$

(b) [9 points]



$$dV = \underbrace{\frac{1}{4\pi\varepsilon_0}}_{k} \cdot \frac{\rho_e \, dz'}{R} \quad , \text{ where } R = \sqrt{\underset{4}{\overset{}{y_p}}^2 + z'^2}$$

$$\Rightarrow dV = k \frac{|z'| \times 10^{-9} \, dz'}{\sqrt{y_p^2 + z'^2}}$$

$$\Rightarrow V = k \cdot 10^{-9} \int_{-2}^{2} \frac{|z'| \, dz'}{\sqrt{y_p^2 + z'^2}} = 2k \cdot 10^{-9} \int_0^2 \frac{z' \, dz'}{\sqrt{y_p^2 + z'^2}}$$

even function of z'

$$\Rightarrow V \approx 2 \times \cancel{9 \times 10^{9}} \times \cancel{10^{-9}} \times \sqrt{y_P^2 + z'^2}\,\Big|_0^2$$

$$V(y_P) \approx 18\left(\sqrt{y_P^2+4} - |y_P|\right)$$

Since $y_P = 4$,

$$V(P) \approx 18\left(\sqrt{16+4} - 4\right) \approx \underline{8.5\ V}$$

(c) [14 points]

<u>Method #1:</u>

$$\vec{E} = -\nabla V$$

Due to symmetry, at $P(0,4,0)$, $\vec{E}$ has only a y-component:

$$\vec{E} = E_y\,\hat{a}_y$$

$$E_y = -\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}\left[18\left(\sqrt{y^2+4} + \underbrace{|y|}_{=y \text{ if } y \geq 0}\right)\right]$$

$$\Rightarrow E_y = -18\left[\frac{1}{\cancel{2}}\frac{\cancel{2}y}{\sqrt{y^2+4}} - 1\right]$$

$$\Rightarrow E_y(0,4,0) = 18\left(1 - \frac{4}{\sqrt{20}}\right) \approx 1.9\ V/m$$

$$\Rightarrow \underline{\vec{E}(P) \approx 1.9\,\hat{a}_y,\ V/m}$$

Method #2 (superposition):

$$d\vec{E} = \underbrace{\frac{1}{4\pi\varepsilon_0}}_{k} \cdot \frac{\rho_\ell \, dz'}{R^3} \vec{R} \quad , \text{where } \vec{R} = y_P \hat{a}_y - z'\hat{a}_z$$

$$\Rightarrow d\vec{E} = k \frac{|z'| \cdot 10^{-9} dz' \cdot (y_P \hat{a}_y - z'\hat{a}_z)}{(y_P^2 + z'^2)^{3/2}}$$

$$\Rightarrow \vec{E} = \hat{a}_y \underbrace{\int_{-2}^{2} k \cdot \frac{y_P |z'| \cdot 10^{-9} dz'}{(y_P^2+z'^2)^{3/2}}}_{\neq 0 \text{ even function of } z'} - \hat{a}_z \underbrace{\int_{-2}^{2} k \cdot \frac{z'|z'| \cdot 10^{-9} dz'}{(y_P^2+z'^2)^{3/2}}}_{0 \text{ odd function of } z'}$$

$$\Rightarrow \vec{E} = \hat{a}_y k \cdot y_P \cdot 10^{-9} \cdot 2 \cdot \int_0^2 \frac{z' dz'}{(y_P^2+z'^2)^{3/2}}$$

$$\Rightarrow \vec{E} \approx \hat{a}_y y_P \cdot 9 \times \cancel{10^9} \times \cancel{10^{-9}} \times 2 \times \left[-\frac{1}{\sqrt{y_P^2+z'^2}} \right] \Big|_0^2$$

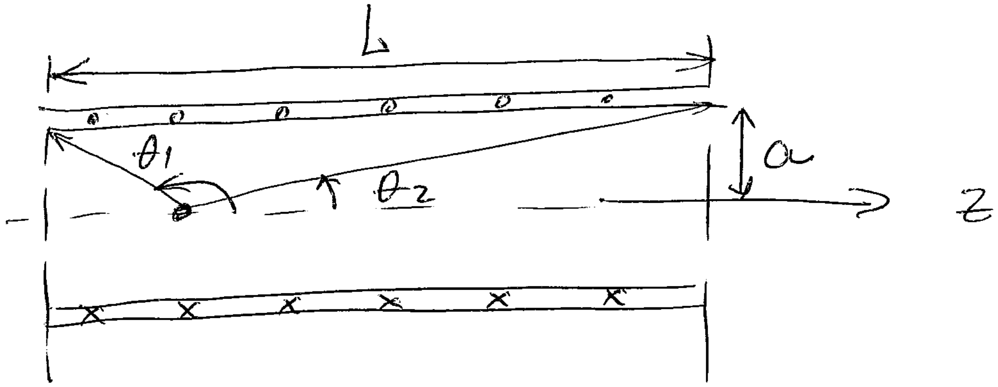
$$\Rightarrow \vec{E} \approx \hat{a}_y y_P \, 18 \left[\frac{1}{|y_P|} - \frac{1}{\sqrt{y_P^2+4}} \right]$$

$$\Rightarrow \vec{E}(P) \approx \hat{a}_y \times 4 \times 18 \times \left(\frac{1}{4} - \frac{1}{\sqrt{20}} \right) \approx \hat{a}_y \, 1.9 \quad V/m$$

[18 points]

(3) $\vec{H} = \underbrace{\frac{nI}{2}(\cos\theta_2 - \cos\theta_1)}_{H_z}\,\hat{a}_z$, where

$n = \frac{N}{L} = \frac{2000}{0.75} \approx 2666.6$



Center: $\cos\theta_2 = \frac{L}{2\sqrt{(L/2)^2 + a^2}} = \frac{1}{\sqrt{1+\left(\frac{2a}{L}\right)^2}}$

$\Rightarrow \cos\theta_2 = \frac{1}{\sqrt{1+\frac{10}{75}}} \approx 0.99123$

$\cos\theta_1 = -\cos\theta_2$

$\Rightarrow H_z = \frac{NI}{\cancel{2}L} \times \cancel{2} \times \cos\theta_2 = \frac{NI}{L}\cos\theta_2$

$\Rightarrow H_z = \frac{2000 \times 0.05}{0.75} \cdot 0.99123 \approx \underline{132.16, \text{A/m}}$

Edge: $\theta_1 = 90^\circ \Rightarrow \cos\theta_1 = 0$

$\cos\theta_2 = \frac{L}{\sqrt{L^2 + a^2}} \approx 0.9978$

$H_z = \frac{NI}{2L} \cdot \cos\theta_2 = \frac{2000 \times 0.05}{2 \times 0.75} \times 0.9978$

$\Rightarrow H_z \approx 66.52$, A/m

(4) [18 points]

$$\vec{A} = (3y - z)\hat{a}_x + 2xz\,\hat{a}_y \quad , \quad Wb/m$$

(a) $\vec{B} = \nabla \times \vec{A}$, where

$$A_x = 3y - z$$
$$A_y = 2xz$$
$$A_z = 0$$

$$\vec{B} = \left(\underbrace{\frac{\partial A_z}{\partial y}}_{=\emptyset} - \frac{\partial A_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial A_x}{\partial z} - \underbrace{\frac{\partial A_z}{\partial x}}_{=\emptyset}\right)\hat{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{a}_z$$

$$\Rightarrow \vec{B} = -2x\,\hat{a}_x - 1\,\hat{a}_y + (2z - 3)\,\hat{a}_z \quad , \quad T$$

$$\Rightarrow \vec{B}(2,-1,3) = -4\hat{a}_x - 1\hat{a}_y + 3\hat{a}_z \; , \; T$$

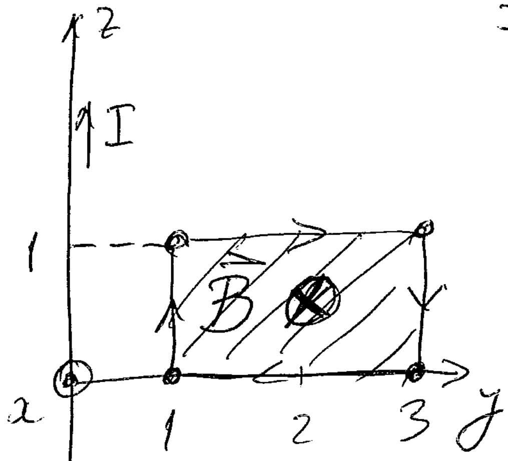
(b) $$\vec{H} = \frac{\vec{B}}{\mu_0} \approx 795.8 \times 10^3 \left[-2x\,\hat{a}_x - 1\,\hat{a}_y + (2z-3)\hat{a}_z\right]$$

$$\Rightarrow \vec{H}(2,-1,3) \approx 10^6\left[-3.183\,\hat{a}_x - 0.796\,\hat{a}_y + 2.387\,\hat{a}_z\right]$$

(c) $$\vec{J} = \nabla \times \vec{H} = \left(\underbrace{\frac{\partial H_z}{\partial y}}_{=0} - \underbrace{\frac{\partial H_y}{\partial z}}_{=0}\right)\hat{a}_x + \left(\underbrace{\frac{\partial H_x}{\partial z}}_{=0} - \underbrace{\frac{\partial H_z}{\partial x}}_{=0}\right)\hat{a}_y + \left(\underbrace{\frac{\partial H_y}{\partial x}}_{=0} - \underbrace{\frac{\partial H_x}{\partial y}}_{=0}\right)\hat{a}_z$$ where

$$H_x = -2x \times 795.8 \times 10^3$$
$$H_y = -795.8 \times 10^3$$
$$H_z = (2z-3)\,795.8 \times 10^3$$

$$\boxed{\vec{J} = 0}$$



(5) [9 points]

$$H = \frac{I}{2\pi\rho} = \frac{I}{2\pi y}$$

$$\Phi = \int_1^3 \int_0^1 \mu_0 |H_x| \, dz \, dy$$

$$\Rightarrow \Phi = \underset{\substack{\| \\ 4\pi \times 10^{-7}}}{\mu_0} \frac{I}{2\pi} \ln 3 = 2 \ln 3 \times 10^{-7} \times \overset{\substack{2 \\ \|}}{I}$$

$$\Rightarrow \Phi \approx 4.4 \times 10^{-7} \text{ Wb}$$

---

(6) [4 points]

$$M = \frac{N \cdot \Phi}{I} = \underset{\substack{\| \\ 150}}{N} \times \mu_0 \frac{\ln 3}{2\pi} \times \frac{I}{I} = 150 \times 2 \times \ln 3 \times 10^{-7}$$

$$\Rightarrow M \approx 330 \times 10^{-7} \text{ H} \approx 33 \ \mu\text{H}$$