EE750
Advanced Engineering Electromagnetics
Lecture 12

Duality

- Duality means that two differential/integral equations describing the behavior of two different variables have the same mathematical forms $\Rightarrow$ solutions are identical

- Equations describing the case ($\boldsymbol{J} \neq \mathbf{0}$, $\boldsymbol{M}=\mathbf{0}$) are dual to equations describing the case ($\boldsymbol{J}=\mathbf{0}$, $\boldsymbol{M} \neq \mathbf{0}$)

$$\nabla \times \boldsymbol{E}_A = -j\omega\mu\, \boldsymbol{H}_A$$

$$\nabla \times \boldsymbol{H}_A = \boldsymbol{J} + j\omega\varepsilon\, \boldsymbol{E}_A$$

$$\nabla^2 \boldsymbol{A} + \beta^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

$$\boldsymbol{A} = \frac{\mu}{4\pi} \iiint_{V'} \frac{\boldsymbol{J}}{R} e^{-j\beta R}\, dV'$$

$$\nabla \times \boldsymbol{H}_F = j\omega\varepsilon\, \boldsymbol{E}_F$$

$$\nabla \times \boldsymbol{E}_F = -\boldsymbol{M} - j\omega\mu\, \boldsymbol{H}_F$$

$$\nabla^2 \boldsymbol{F} + \beta^2 \boldsymbol{F} = -\varepsilon \boldsymbol{M}$$

$$\boldsymbol{F} = \frac{\varepsilon}{4\pi} \iiint_{V'} \frac{\boldsymbol{M}}{R} e^{-j\beta R}\, dV'$$

Duality (Cont'd)

$$\boldsymbol{H}_A = (1/\mu)\nabla\times\boldsymbol{A}$$

$$\boldsymbol{E}_A = -j\omega\boldsymbol{A} - (j/\omega\mu\varepsilon)(\nabla(\nabla.\boldsymbol{A}))$$

$$\boldsymbol{E}_F = (-1/\varepsilon)\nabla\times\boldsymbol{F}$$

$$\boldsymbol{H}_F = -j\omega\boldsymbol{F} - (j/\omega\mu\varepsilon)(\nabla(\nabla.\boldsymbol{F}))$$

It follows that the following quantities are identical

$\boldsymbol{E}_A \Leftrightarrow \boldsymbol{H}_F$ $\quad \boldsymbol{H}_A \Leftrightarrow -\boldsymbol{E}_F$ $\quad \boldsymbol{J} \Leftrightarrow \boldsymbol{M}$

$\boldsymbol{A} \Leftrightarrow \boldsymbol{F}$ $\quad \varepsilon \Leftrightarrow \mu$ $\quad \mu \Leftrightarrow \varepsilon$

$\beta \Leftrightarrow \beta$ $\quad \eta \Leftrightarrow 1/\eta$

Example

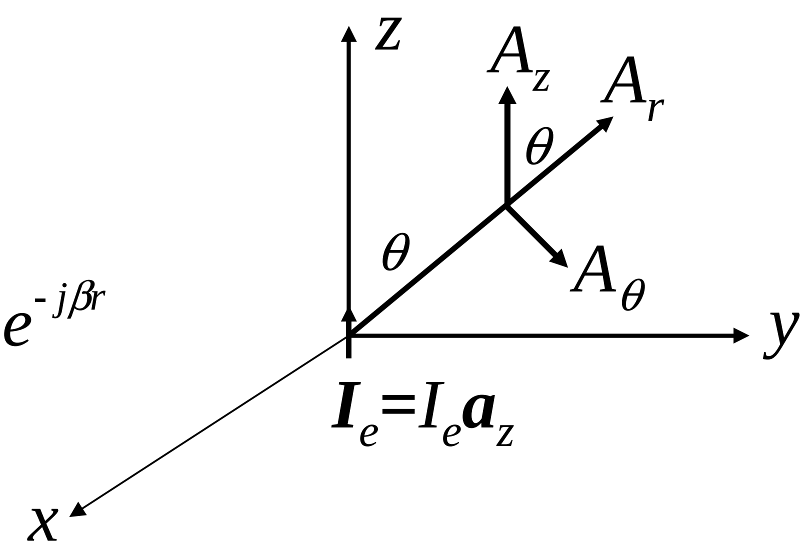
Using Duality, find the fields resulting from an infinitesimal magnetic dipole $\boldsymbol{I}_m=\boldsymbol{a}_z I_m$

The fields resulting from an <u>electric</u> dipole are

$$E_r = \frac{\eta I_e l}{2\pi r^2} cos\theta (1+\frac{1}{j\beta r}) e^{-j\beta r}$$

$$E_\theta = \frac{j\eta\beta I_e l}{4\pi r} sin\theta (1+\frac{1}{j\beta r}-\frac{1}{(\beta r)^2}) e^{-j\beta r}$$

$$H_\varphi = \frac{j\beta I_e l}{4\pi r} sin\theta (1+\frac{1}{j\beta r}) e^{-j\beta r}$$



Example (Cont'd)

It follows that the fields resulting from the magnetic dipole are given by

$$H_r = \frac{I_m l}{2\eta\pi r^2} cos\theta(1 + \frac{1}{j\beta r}) e^{-j\beta r}$$

$$H_\theta = \frac{j\beta I_m l}{4\eta\pi r} sin\theta(1 + \frac{1}{j\beta r} - \frac{1}{(\beta r)^2}) e^{-j\beta r}$$

$$E_\varphi = \frac{-j\beta I_m l}{4\pi r} sin\theta(1 + \frac{1}{j\beta r}) e^{-j\beta r}$$

Uniqueness Theorem

- This theorem establishes the conditions under which a unique solution exists for a given problem
- Assume that a closed surface S encloses a material with sources $\boldsymbol{J}_i$, $\boldsymbol{M}_i$ and complex parameters $\varepsilon = \varepsilon' - j\varepsilon''$, $\mu = \mu' - j\mu''$
- If there are two possible solutions $\boldsymbol{E}^a$, $\boldsymbol{H}^a$ and $\boldsymbol{E}^b$, $\boldsymbol{H}^b$, they must satisfy Maxwell's equations

$$\nabla \times \boldsymbol{E}^a = -\boldsymbol{M}_i - j\omega\mu\, \boldsymbol{H}^a \quad , \qquad \nabla \times \boldsymbol{H}^a = \boldsymbol{J}_i + \sigma\, \boldsymbol{E}^a + j\omega\varepsilon\, \boldsymbol{E}^a$$

$$\nabla \times \boldsymbol{E}^b = -\boldsymbol{M}_i - j\omega\mu\, \boldsymbol{H}^b \quad , \qquad \nabla \times \boldsymbol{H}^b = \boldsymbol{J}_i + \sigma\, \boldsymbol{E}^b + j\omega\varepsilon\, \boldsymbol{E}^b$$

Uniqueness Theorem (Cont'd)

- Subtracting the corresponding equations we get

  $$\nabla\times\delta\boldsymbol{E} = -j\omega\mu\delta\boldsymbol{H}\ , \qquad \nabla\times\delta\boldsymbol{H} = (\sigma + j\omega\varepsilon)\delta\boldsymbol{E}$$

  where $\delta\boldsymbol{E}=\boldsymbol{E}^a-\boldsymbol{E}^b$ and $\delta\boldsymbol{H}=\boldsymbol{H}^a-\boldsymbol{H}^b$

- Notice that the differential fields satisfy the source-free Maxwell's equations

- Applying the source-free conservation of energy relation for $\delta\boldsymbol{E}$ and $\delta\boldsymbol{H}$ we get

$$\oiint_S (\delta\boldsymbol{E}\times\delta\boldsymbol{H}^*).\boldsymbol{ds} = -\iiint_V \left[\delta\boldsymbol{E}.(\sigma + j\omega\varepsilon)\,\delta\boldsymbol{E}^* + \delta\boldsymbol{H}^*.(j\omega\mu)\delta\boldsymbol{H}\right] dv$$

$$\Downarrow$$

$$\oiint_S (\delta\boldsymbol{E}\times\delta\boldsymbol{H}^*).\boldsymbol{ds} = -\iiint_V ((\sigma + \omega\varepsilon'')\left|\delta\boldsymbol{E}\right|^2 + \omega\mu''\left|\delta\boldsymbol{H}\right|^2)\, dv$$

$$- j\iiint_V (-\omega\varepsilon'\left|\delta\boldsymbol{E}\right|^2 + \omega\mu'\left|\delta\boldsymbol{H}\right|^2)\, dv$$

Uniqueness Theorem (Cont'd)

- Now if we have $\oiint_S (\delta\boldsymbol{E} \times \delta\boldsymbol{H}^*).\boldsymbol{ds} = 0$, this implies that $\delta\boldsymbol{E}=\delta\boldsymbol{H}=\mathbf{0}$ everywhere inside *S*. Notice that the assumption of losses existence is important!

- Using the vector identity $\boldsymbol{A}\bullet\boldsymbol{B}\times\boldsymbol{C}= \boldsymbol{B}\bullet\boldsymbol{C}\times\boldsymbol{A}= \boldsymbol{C}\bullet\boldsymbol{A}\times\boldsymbol{B}$, we have

$$\oiint_S (\delta\boldsymbol{E} \times \delta\boldsymbol{H}^*).ds\boldsymbol{n} = \oiint_S (\boldsymbol{n}\times\delta\boldsymbol{E}).\,\delta\boldsymbol{H}^*\, ds = \oiint_S (\delta\boldsymbol{H}^* \times \boldsymbol{n}).\delta\boldsymbol{E}\,ds$$

- It follows that the condition $\oiint_S (\delta\boldsymbol{E} \times \delta\boldsymbol{H}^*).\boldsymbol{ds} = 0$ implies that one of the following three cases is satisfied:

a) The tangential component of the ***E*** field is specified on *S*, i.e. $\boldsymbol{n} \times\delta\boldsymbol{E} =\mathbf{0}$ on *S*

Uniqueness Theorem (Cont'd)

b) The tangential component of the ***H*** field is specified on *S*, i.e. $\boldsymbol{n}\times\delta\boldsymbol{H}=\boldsymbol{0}$ on S $\Longrightarrow$ $\delta\boldsymbol{H}^{*}\times\boldsymbol{n}=\boldsymbol{0}$ on S

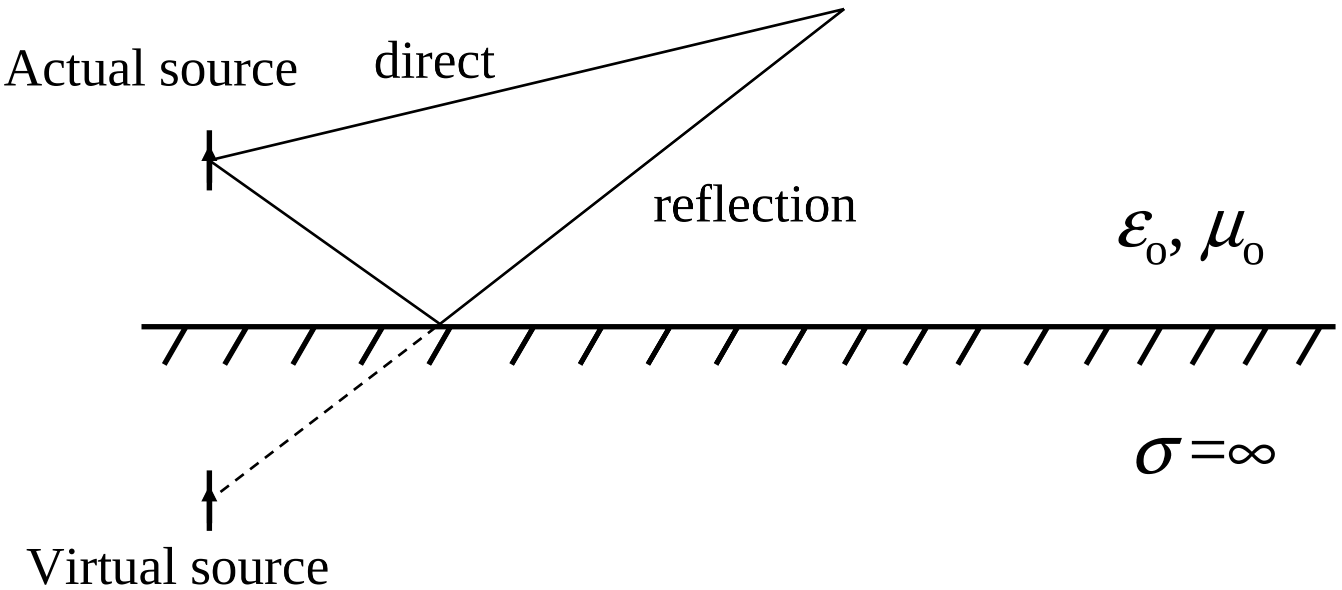
c) The tangential component of the ***E*** field is specified on part of *S* and the tangential component of the ***H*** field is specified on the rest of *S*, i.e.

$$\boldsymbol{n}\times\delta\boldsymbol{E}=\boldsymbol{0} \text{ on } S_1$$

$$\boldsymbol{n}\times\delta\boldsymbol{H}=\boldsymbol{0} \text{ on } S_2$$

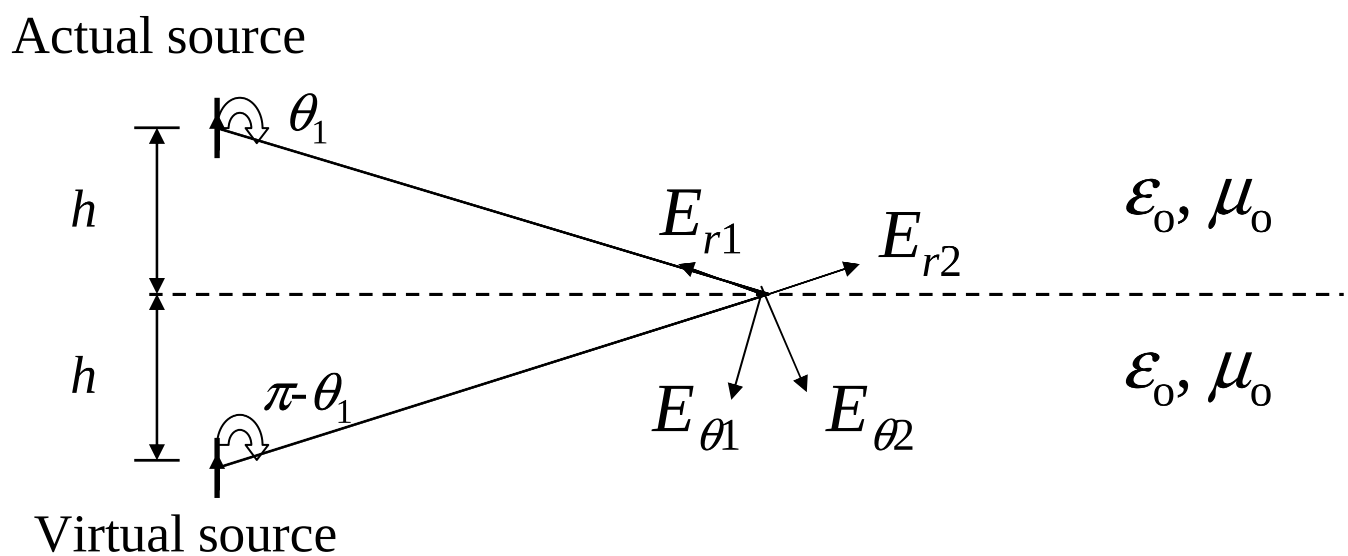
$$S=S_1\cup S_2$$

Image Theory



- Image theory enables us to evaluate the field generated by sources placed near infinite perfectly conducting boundary
- Virtual sources are added to maintain the same tangential field boundary conditions for the original problem

Image Theory (Cont'd)



- Image of a vertical electric dipole is another vertical electric dipole (same orientation)
- Notice that the tangential electric field components cancel out

Image Theory (Cont'd)

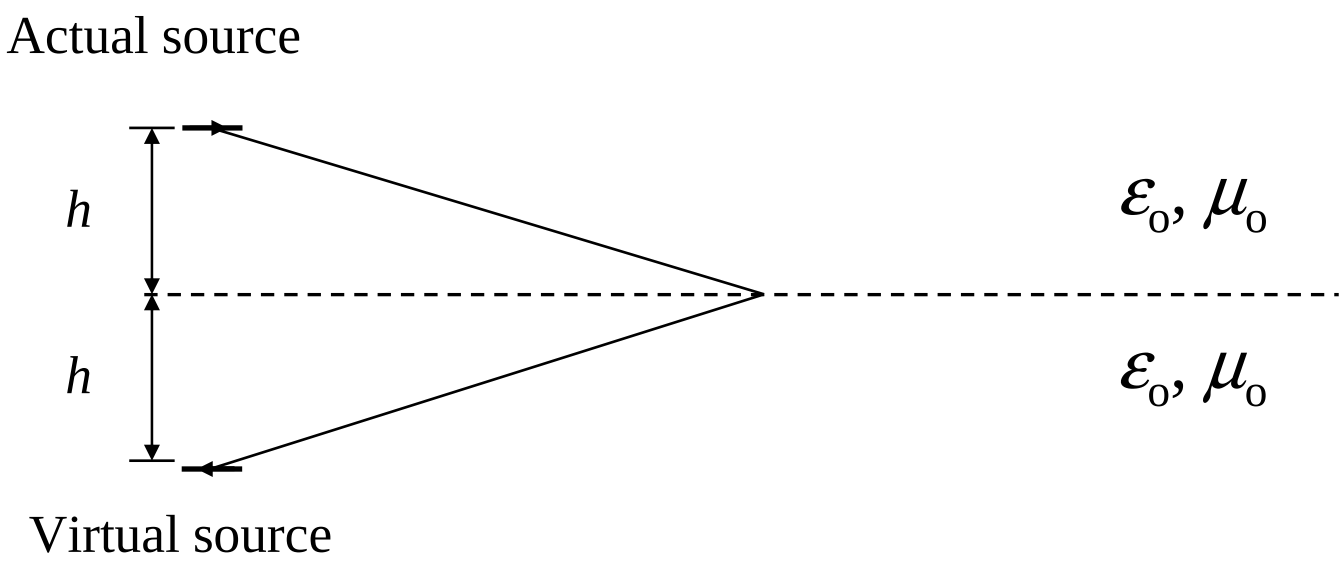
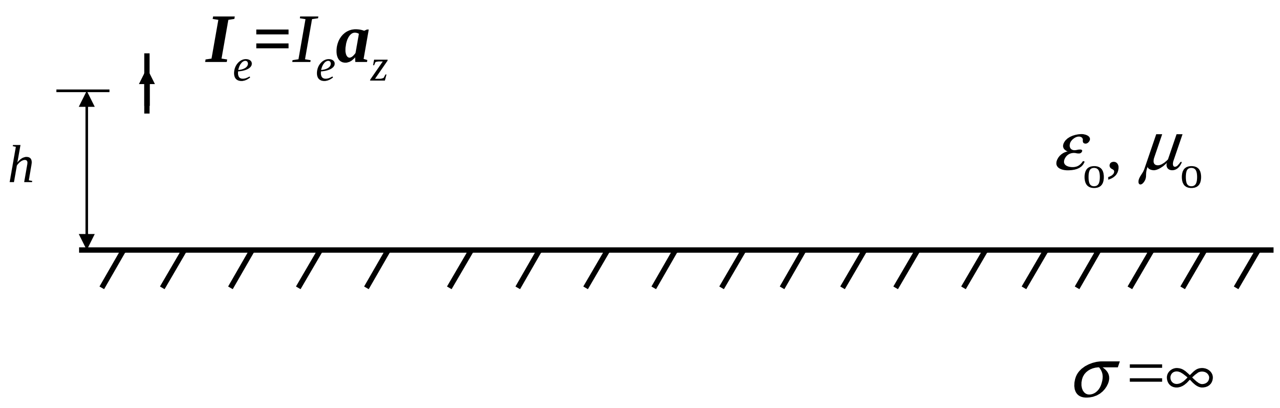


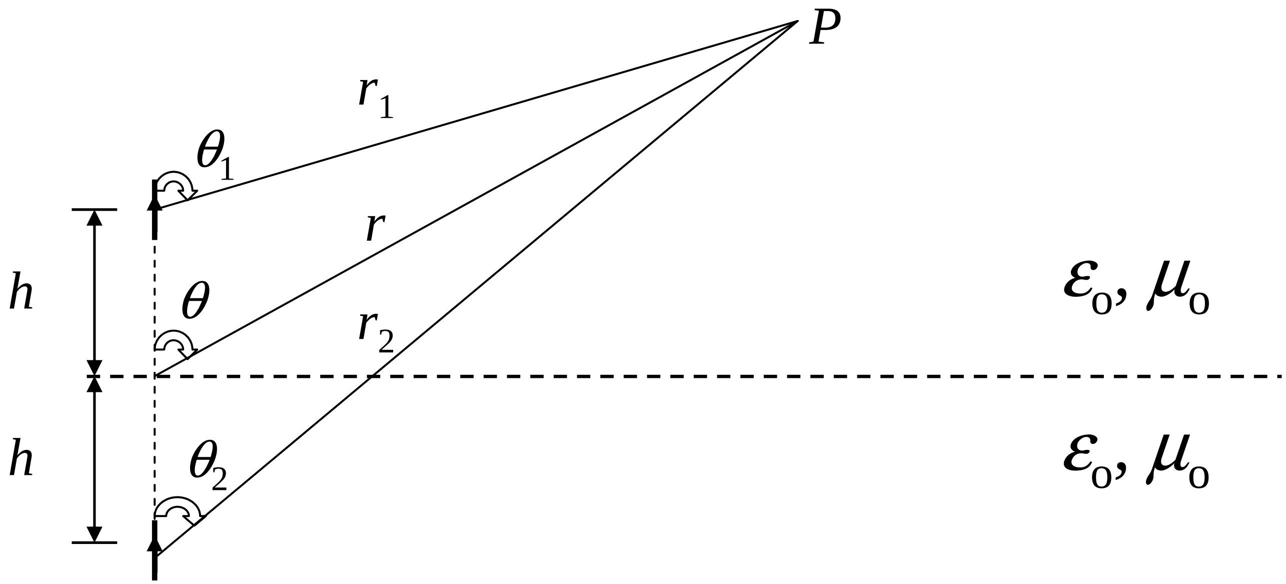
Image for a horizontal electric dipole has the same value but opposite orientation (Prove it!)

Example



- Obtain an expression for the far fields generated by a vertical electric dipole of length *l* placed near an infinite conducting wall

Example (Cont'd)



For the far fields we have

$$E_\theta^1 = \frac{j\eta\beta I_e\, l}{4\pi\, r_1} sin\,\theta_1 e^{-j\beta r_1} \quad \text{and} \quad E_\theta^2 = \frac{j\eta\beta I_e\, l}{4\pi\, r_2} sin\,\theta_2 e^{-j\beta r_2}$$

$$r_1^2 = r^2 + h^2 - 2\, r\, h\, cos\theta$$

$$r_2^2 = r^2 + h^2 + 2\, r\, h\, cos\theta$$

$r >> h \Rightarrow$

$$r_1 = r - h\cos\theta$$

$$r_2 = r + h\cos\theta$$

Example (Cont'd)

- Further, we can use for the amplitude $r=r_1=r_2$
- The total field in the top half space is the sum of the field generated by both the actual and virtual sources

$$E_\theta = E_\theta^1 + E_\theta^2 = \frac{j\eta\beta I_e\, l}{4\pi r} \sin\theta\, e^{-j\beta r}\left(e^{j\beta h\cos\theta} + e^{-j\beta h\cos\theta}\right)$$

$z \geq 0$

$E_\theta=0, \quad z<0$

- Combining terms we get

$$E_\theta = \underbrace{\frac{j2\eta\beta I_e\, l}{4\pi r} \sin\theta\, e^{-j\beta r}}_{\text{element factor}} \underbrace{\cos(\beta h\cos\theta)}_{\text{array factor}}$$

Reciprocity Theorem

- Reciprocity theorem in circuit theory states that we can change the location of the source and observation points without affecting the measured values
- A similar theorem can be derived for electromagnetics
- Assume that two sets of sources $\boldsymbol{J}_1$, $\boldsymbol{M}_1$ and $\boldsymbol{J}_2$, $\boldsymbol{M}_2$ radiate within a linear isotropic medium
- Using Maxwell's equations, we have

$$\nabla \times \boldsymbol{E}_1 = -\boldsymbol{M}_1 - j\omega\mu \boldsymbol{H}_1 \qquad \nabla \times \boldsymbol{E}_2 = -\boldsymbol{M}_2 - j\omega\mu \boldsymbol{H}_2$$

$$\nabla \times \boldsymbol{H}_1 = \boldsymbol{J}_1 + j\omega\varepsilon \boldsymbol{E}_1 \qquad \nabla \times \boldsymbol{H}_2 = \boldsymbol{J}_2 + j\omega\varepsilon \boldsymbol{E}_2$$

Reciprocity Theorem (Cont'd)

- Dot multiplying the $\boldsymbol{H}_2$ curl equation by $\boldsymbol{E}_1$ and dot multiplying the $\boldsymbol{E}_1$ curl equation by $\boldsymbol{H}_2$ and subtracting we get

$$\boldsymbol{E}_1.\nabla\times\boldsymbol{H}_2-\boldsymbol{H}_2.\nabla\times\boldsymbol{E}_1=\boldsymbol{E}_1.\boldsymbol{J}_2+\boldsymbol{H}_2.\boldsymbol{M}_1+j\omega\varepsilon\,\boldsymbol{E}_1.\boldsymbol{E}_2+j\omega\mu\,\boldsymbol{H}_1.\boldsymbol{H}_2$$

$$\Downarrow$$

$$-\nabla.(\boldsymbol{E}_1\times\boldsymbol{H}_2)=\boldsymbol{E}_1.\boldsymbol{J}_2+\boldsymbol{H}_2.\boldsymbol{M}_1+j\omega\varepsilon\,\boldsymbol{E}_1.\boldsymbol{E}_2+j\omega\mu\,\boldsymbol{H}_1.\boldsymbol{H}_2$$

- Similarly, we obtain

$$-\nabla.(\boldsymbol{E}_2\times\boldsymbol{H}_1)=\boldsymbol{E}_2.\boldsymbol{J}_1+\boldsymbol{H}_1.\boldsymbol{M}_2+j\omega\varepsilon\,\boldsymbol{E}_2.\boldsymbol{E}_1+j\omega\mu\,\boldsymbol{H}_2.\boldsymbol{H}_1$$

- Subtracting these two expressions, we obtain

$$-\nabla.(\boldsymbol{E}_1\times\boldsymbol{H}_2-\boldsymbol{E}_2\times\boldsymbol{H}_1)=\boldsymbol{E}_1.\boldsymbol{J}_2+\boldsymbol{H}_2.\boldsymbol{M}_1-\boldsymbol{E}_2.\boldsymbol{J}_1-\boldsymbol{H}_1.\boldsymbol{M}_2$$

Reciprocity Theorem (Cont'd)

- Applying divergence theorem, we obtain the Lorentz reciprocity theorem

$$-\oiint_S (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) . \boldsymbol{ds} = \iiint_V (\boldsymbol{E}_1 . \boldsymbol{J}_2 + \boldsymbol{H}_2 . \boldsymbol{M}_1 - \boldsymbol{E}_2 . \boldsymbol{J}_1 - \boldsymbol{H}_1 . \boldsymbol{M}_2) dV$$

- For a source-free region we have

$$-\oiint_S (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \bullet \boldsymbol{ds} = 0$$

- If S is taken as a sphere of infinite radius, we have

$$\iiint_V (\boldsymbol{E}_1 . \boldsymbol{J}_2 + \boldsymbol{H}_2 . \boldsymbol{M}_1 - \boldsymbol{E}_2 . \boldsymbol{J}_1 - \boldsymbol{H}_1 . \boldsymbol{M}_2) dV = 0$$

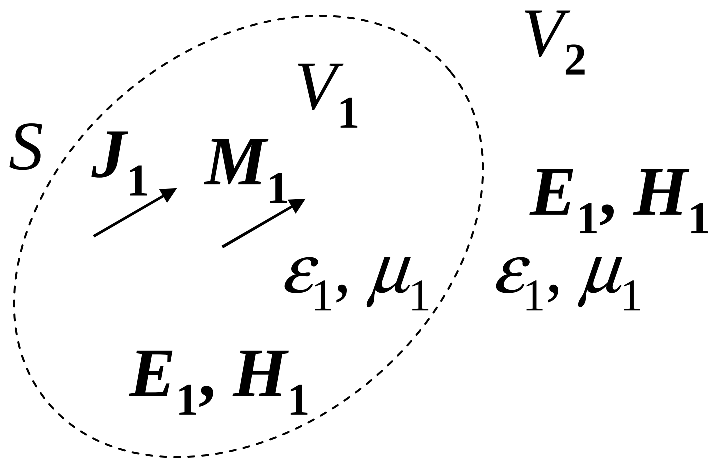
$$\Downarrow$$

$$\iiint_V (\boldsymbol{E}_1 . \boldsymbol{J}_2 - \boldsymbol{H}_1 . \boldsymbol{M}_2) dV = \iiint_V (\boldsymbol{E}_2 . \boldsymbol{J}_1 - \boldsymbol{H}_2 . \boldsymbol{M}_1) dV$$

Surface Equivalence Theorem

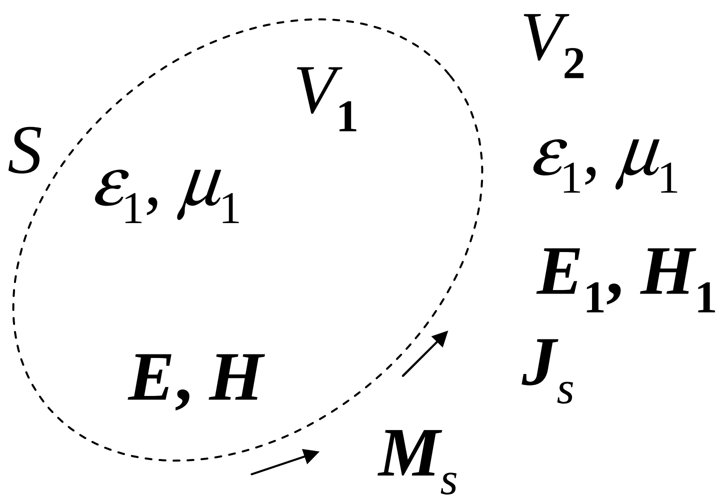
- This theorem is based on the uniqueness theorem
- It obtains the fields outside an imaginary surface by placing electric and magnetic sources on the boundary so that the same boundary conditions are satisfied
- Assume that sources $\boldsymbol{J}_1$ and $\boldsymbol{M}_1$ radiate in an unbounded medium
- We place a virtual surface S that encloses these sources

Surface Equivalence Theorem (Cont'd)



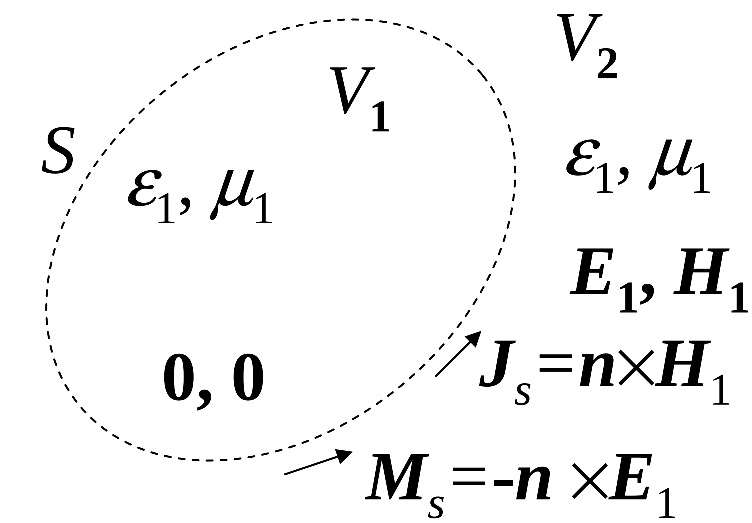
Remove the sources and assume arbitrary fields ***E***, ***H*** inside *S*. For the boundary conditions to be satisfied we add the boundary sources $\boldsymbol{J}_s = \boldsymbol{n} \times (\boldsymbol{H}_1 - \boldsymbol{H})$

$$\boldsymbol{M}_s = -\boldsymbol{n} \times (\boldsymbol{E}_1 - \boldsymbol{E})$$

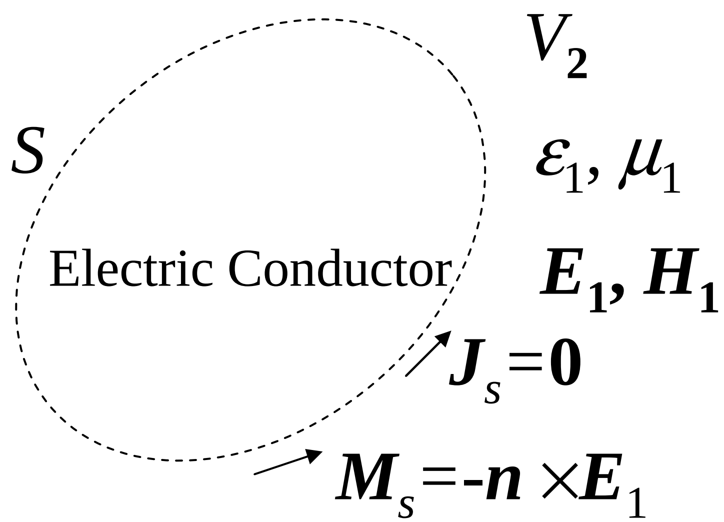


As ***E*** and ***H*** are chosen arbitrarily we may choose ***E*=0, *H*=0**

Surface Equivalence Theorem (Cont'd)

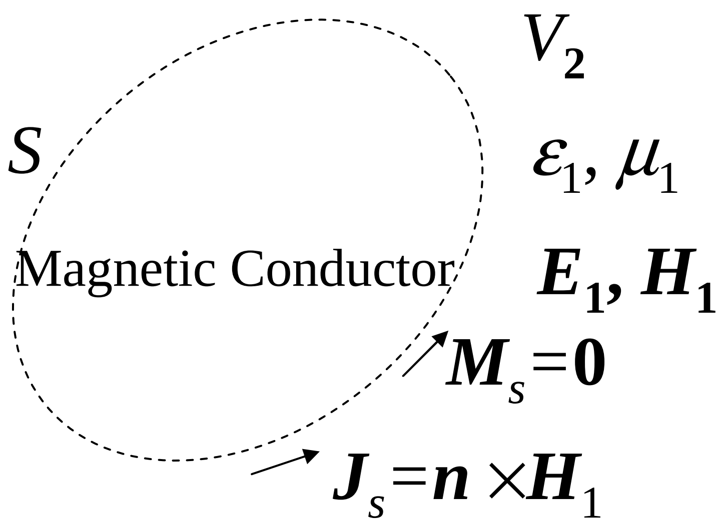


As $\boldsymbol{J}_s$ and $\boldsymbol{M}_s$ represent the tangential components of the $\boldsymbol{H}_1$ and $\boldsymbol{E}_1$ fields, only one of them is needed according to uniqueness theorem



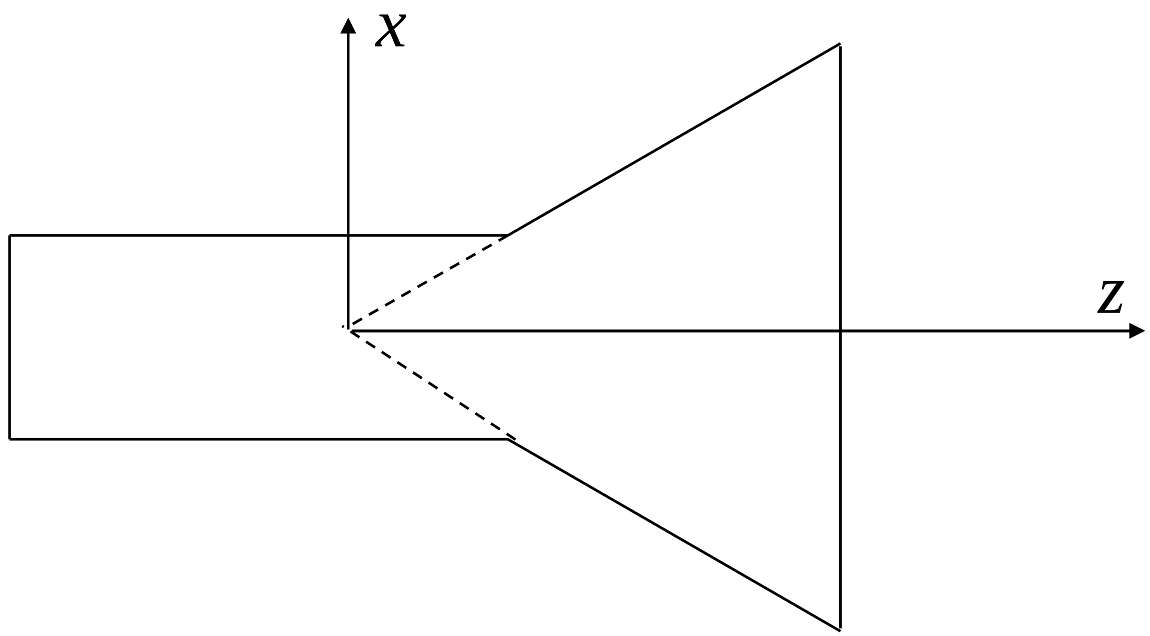
Notice that $\boldsymbol{J}_s$ is shorted out

Surface Equivalence Theorem (Cont'd)



Notice that $\boldsymbol{M}_s$ is shorted out

Application: H-plan Horn Antenna



- At the surface we know E_y and H_x
- Equivalent sources are given by $J_y=H_x$ and $M_x=E_y$