EE750
Advanced Engineering Electromagnetics
Lecture 15

The Finite Element Method (FEM)

- The Ritz Method
- Galerkin's Method
- Introduction to FEM and general steps

References

Jianming Jin, The Finite Element Method in Electromagnetics, 2nd edition, John Wiley & Sons, Inc.

M. Sadiku, Numerical Techniques in Electromagnetics, CRC press

John L. Volakis, Arindam Chatterjee, and Leo C. Kempel, Finite Element Method for Electromagnetics, IEEE Press.

Ritz Method

- This method aims at solving a Boundary Value Problem (BVP) of the form $L(\Phi)=f$, by minimizing a corresponding functional $F(\Phi)$
- Example: Solve the BVP $\frac{d^2\varphi}{dx^2}=x+1$ $\quad 0< x <1$

  subject to $\varphi(0)=0,\quad \varphi(1)=1$ using Ritz method
- We define the corresponding functional

$$F(\varphi)=0.5\int_0^1\left(\frac{d\varphi}{dx}\right)^2 dx+\int_0^1 (x+1)\varphi dx$$

- Notice that for every possible trial function $\tilde{\varphi}(x)$ the functional F assumes a certain value $F(\tilde{\varphi})$

Ritz Method (Cont'd)

- We want to show that the minimum of F is assumed at a function $\tilde{\varphi} = \varphi_s^*$, the solution of the BVP
- If a trial function φ is perturbed by a function $\delta\varphi$, the functional F changes by ΔF where

$$\Delta F = F(\varphi + \delta\varphi) - F(\varphi) = \delta F + O(\delta\varphi^2)$$

$$F(\varphi + \delta\varphi) = 0.5\int_0^1 \left(\frac{d(\varphi + \delta\varphi)}{dx} \right)^2 dx + \int_0^1 (x+1)(\varphi + \delta\varphi)\, dx$$

$$\Downarrow$$

$$F(\varphi + \delta\varphi) = 0.5\int_0^1 \left(\frac{d\varphi}{dx} \right)^2 dx + \int_0^1 \left(\frac{d\varphi}{dx} \right) \left(\frac{d\delta\varphi}{dx} \right) dx$$

$$+ 0.5\int_0^1 \left(\frac{d\delta\varphi}{dx} \right)^2 dx + \int_0^1 (x+1)\varphi\, dx + \int_0^1 (x+1)\delta\varphi\, dx$$

Ritz Method (Cont'd)

- If follows that we have

$$\Delta F = \int_0^1 \left(\frac{d(\varphi)}{dx} \right) \left(\frac{d(\delta\varphi)}{dx} \right) dx + 0.5 \int_0^1 \left(\frac{d(\delta\varphi)}{dx} \right)^2 dx + \int_0^1 (x+1)\delta\varphi \, dx$$

$$\Downarrow$$

$$\delta F = \int_0^1 \left(\frac{d(\varphi)}{dx} \right) \left(\frac{d(\delta\varphi)}{dx} \right) dx + \int_0^1 (x+1)\delta\varphi \, dx$$

- For optimality, we should have $\delta F=0$

$$\int_0^1 \left(\frac{d(\varphi)}{dx} \right) \left(\frac{d(\delta\varphi)}{dx} \right) dx + \int_0^1 (x+1)\delta\varphi \, dx = 0$$

$\Downarrow$ Integrate by parts

$$\delta\varphi \left. \frac{d\varphi}{dx} \right|_0^1 - \int_0^1 \frac{d^2\varphi}{dx^2} \delta\varphi \, dx + \int_0^1 (x+1)\delta\varphi \, dx = 0$$

Ritz Method (Cont'd)

- But as $\delta\varphi(0)= \delta\varphi(1)=0$ because of fixed boundary conditions, the optimality condition gives

$$\int_0^1 \left(\frac{d^2\varphi}{dx^2} - (x+1)\right)\delta\varphi \; dx = 0$$

- optimality condition has to apply for any perturbation $\delta\varphi$, it follows that the minimizer of the functional satisfies,

$$\frac{d^2\varphi}{dx^2} - (x+1) = 0$$ which is our BVP

- The functional *F* was formulated such that its minimizer is the solution of the BVP we wish to solve

Ritz Method (Cont'd)

- If we assume a solution of the form

$$\tilde{\varphi}(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

- Applying the boundary conditions we have c_1=0 and c_2=1-c_3-c_4 $\Rightarrow$ $\tilde{\varphi}(x) = x + c_3(x^2 - x) + c_4(x^3 - x)$
- Substituting into the functional

$$F(\varphi) = 0.5\int_0^1 \left(\frac{d\varphi}{dx}\right)^2 dx + \int_0^1 (x+1)\varphi dx$$

$$\Downarrow$$

$$F(c_3, c_4) = \frac{2}{5}c_4^2 + \frac{1}{6}c_3^2 + \frac{1}{2}c_3 c_4 - \frac{23}{60}c_4 - \frac{1}{4}c_3 + \frac{4}{3}$$

Ritz Method (Cont'd)

- Applying optimality conditions we get

$$\frac{\partial F}{\partial c_3} = \frac{1}{3}c_3 + \frac{1}{2}c_4 - \frac{1}{4} = 0, \qquad \frac{\partial F}{\partial c_4} = \frac{1}{2}c_3 + \frac{4}{5}c_4 - \frac{23}{60} = 0$$

$c_3 = 1/2$, $c_4 = 1/6$ $\Rightarrow$ $\tilde{\varphi}(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x$

General Steps for the Ritz Method

- Formulate a functional whose minimizer is the solution of the BVP
- Apply optimality conditions to determine the parameters of the solution

Galerkin's Method

- This method seeks a solution to the BVP $L(\varphi)=f$ by weighting the residual of the differential equation
- For a trial function $\tilde{\varphi}(x)$ this residual is defined by

$$r = L(\tilde{\varphi}) - f$$

- The unknown solution is expressed as a sum of known entire domain basis functions $\varphi = \sum_i c_i \boldsymbol{v}_i \Rightarrow \varphi = \boldsymbol{v}^T \boldsymbol{c}$

where $\boldsymbol{v} = [v_1 \;\; v_2 \;\; \cdots v_N]^T$ and $\boldsymbol{c} = [c_1 \;\; c_2 \;\; \cdots c_N]^T$

- We define the *i*th weighted residual as

$$R_i = \int_\Omega w_i \, r \, d\Omega, \; i = 1, 2, \cdots, N$$

- We set the weighted residuals to zero to obtain N equations in N unknowns

Galerkin's Method (Cont'd)

- For this method we choose $w_i = v_i$ to have

$$R_i = \int_{\Omega} v_i r \, d\Omega = \int_{\Omega} (v_i L(\mathbf{v}^T)\mathbf{c} - v_i f) \, d\Omega, \quad i = 1, 2, \cdots, N$$

- Example: Solve the BVP $\dfrac{d^2\varphi}{dx^2} = x + 1 \qquad 0 < x < 1$

  subject to $\varphi(0) = 0, \quad \varphi(1) = 1$ using Galerkin's method

- As shown before we selected the trial functions as

$$\tilde{\varphi}(x) = x + c_3(x^2 - x) + c_4(x^3 - x)$$

- The residual for this trial function is $r = 2c_3 + 6c_4 x - x - 1$

- We select as weighting functions $w_1 = (x^2 - x),\ w_2 = (x^3 - x)$

Galerkin's Method (Cont'd)

- The weighted residuals are thus given by

$$R_1 = \int_0^1 (x^2 - x)r\,dx = \frac{c_3}{4} + \frac{c_4}{2} - \frac{1}{4} = 0$$

$$R_2 = \int_0^1 (x^3 - x)r\,dx = \frac{c_3}{2} + \frac{4c_4}{5} - \frac{23}{60} = 0$$

- Solving these two equations we get $c_3=1/2$, $c_4=1/6$

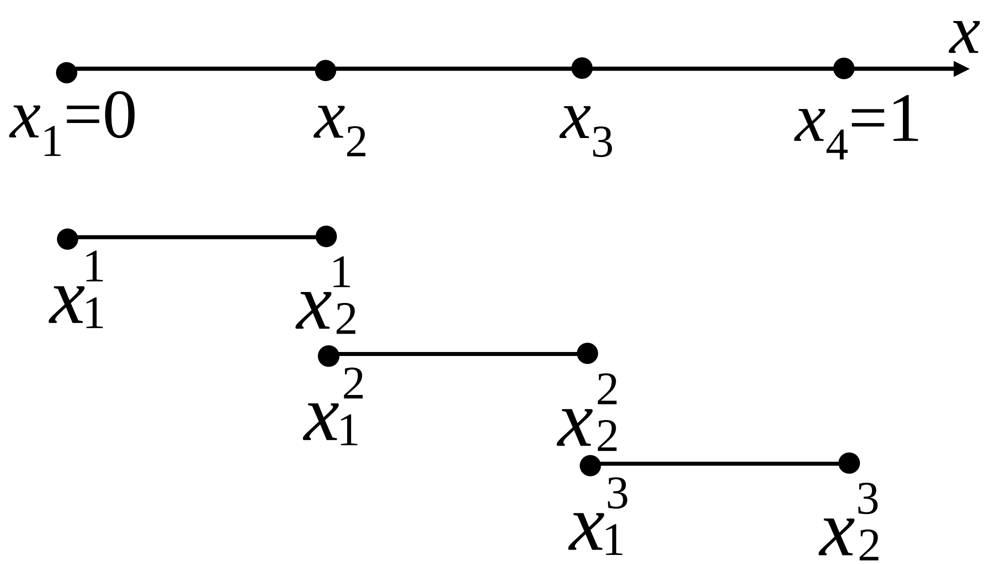
General Steps for Galerkin's Method

- Expand the unknown solution in terms of basis functions
- Evaluate the weighted residuals using the basis functions as weighting functions
- Solve the resultant system of equations for the known coefficients

Introduction to FEM

- We introduce the FEM by solving the previous example

$$\frac{d^2\varphi}{dx^2} = x+1 \;\; , 0< x <1, \text{ subject to } \;\; \varphi(0)=0, \quad \varphi(1)=1$$



- We discretize the space into 3 subdivisions (elements)
- Notice that each node has both a local and a global index, i.e., there are two numbering schemes

Introduction to FEM (Cont'd)

- Over the *i*th element, the unknown function is expressed as an interpolation of the unknown nodes values

$$\varphi(x) = \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)\varphi_i + \left(\frac{x - x_i}{x_{i+1} - x_i}\right)\varphi_{i+1},\ i = 1, 2, 3,\ x_i \le x \le x_{i+1}$$

- Notice that φ_i, i=1, 2, 3, 4 are not known in general. In this problem only the boundary values are known (φ_1=0 and φ_4=1)
- We can formulate FEM using either Ritz's or Galerkin's methods

Introduction to FEM (Cont'd)

- For the Ritz method, we utilize the functional

$$F(\tilde{\varphi}) = 0.5\int_0^1 \left(\frac{d\tilde{\varphi}}{dx}\right)^2 dx + \int_0^1 (x+1)\tilde{\varphi}\,dx$$

$$\Downarrow$$

$$F(\tilde{\varphi}) = \sum_{i=1}^{3}\left(0.5\int_{x_i}^{x_{i+1}} \left(\frac{d\tilde{\varphi}}{dx}\right)^2 dx + \int_{x_i}^{x_{i+1}} (x+1)\tilde{\varphi}\,dx\right)$$

$$\Downarrow$$

$$F = \sum_{i=1}^{3}\left(\begin{array}{l} 0.5\displaystyle\int_{x_i}^{x_{i+1}} \left(\frac{\varphi_{i+1}-\varphi_i}{x_{i+1}-x_i}\right)^2 dx \\ + \displaystyle\int_{x_i}^{x_{i+1}} (x+1)\left(\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)\varphi_i + \left(\frac{x - x_i}{x_{i+1}-x_i}\right)\varphi_{i+1}\right)dx \end{array}\right)$$

Introduction to FEM (Cont'd)

- Integrating, we get

$$F=\sum_{i=1}^{3}\left(\begin{array}{l}0.5(x_{i+1}-x_i)\left(\dfrac{\varphi_{i+1}-\varphi_i}{x_{i+1}-x_i}\right)^2+\varphi_{i+1}\left(\dfrac{2}{3}x_{i+1}+\dfrac{1}{3}x_i+1\right)\\+\varphi_i\left(\dfrac{2}{3}x_i+\dfrac{1}{3}x_{i+1}+1\right)\end{array}\right)$$

⇓ (φ_1=0 and φ_4=1)

$$F=3\varphi_2^2+3\varphi_3^2-3\varphi_2\varphi_3+(4/9)\varphi_2-(22/9)\varphi_3+(49/27)$$

- Applying optimality conditions for the minimizer of F

$$\frac{\partial F}{\partial \varphi_2}=6\varphi_2-3\varphi_3+(4/9)=0$$

$$\frac{\partial F}{\partial \varphi_3}=-3\varphi_2+6\varphi_3-(22/9)=0$$

⇒ φ_2=24/81, φ_3=40/81

Introduction to FEM (Cont'd)

- The same result can be obtained using Galerkin's method with the weighting functions

$$w_i = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, \text{ for } x_{i-1} < x < x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, \text{ for } x_i < x < x_{i+1} \end{cases}$$

Prove it!

- We shall focus on the Ritz finite element method

General Steps of the Ritz FEM

- Divide the domain into subdomains (elements) Ω_e, e=1, 2, …, M
- Over each element, expand the unknown function as an interpolation of the values of the element's nodes $\varphi^e(\boldsymbol{r}) = \sum_{j=1}^{n} N_j^e(\boldsymbol{r})\,\varphi_j^e, \boldsymbol{r} \in \Omega_e$, where φ_j^e is the value of φ at the jth node of the eth element and $N_j^e(\boldsymbol{r})$ is the corresponding interpolation function
- Formulate the functional in terms of the unknown coefficients $F = \sum_{e=1}^{M} F^e(\tilde{\varphi}^e)$
- Apply the optimality conditions for a minimizer of the functional $\partial F / \partial \varphi_i = 0$, i=1,2, …, N
- Solve the resultant system of equations