

**EE750**  
**Advanced Engineering Electromagnetics**  
**Lecture 16**

# 1D FEM

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- We consider a 1D differential equation of the form

$$-\frac{d}{dx}\left(\alpha\frac{d\varphi}{dx}\right) + \beta\varphi = f, \quad x \in (0, L) \quad \text{subject to the}$$

$$\text{boundary conditions } \varphi(0) = p, \quad \left[\alpha\frac{d\varphi}{dx} + \gamma\varphi\right]_{x=L} = q$$

- $\alpha$  and  $\beta$  are functions associated with the physical parameters and  $f$  is the excitation
- Notice that the boundary conditions may be a Dirichlet, Neuman or mixed Dirichlet and Neuman.

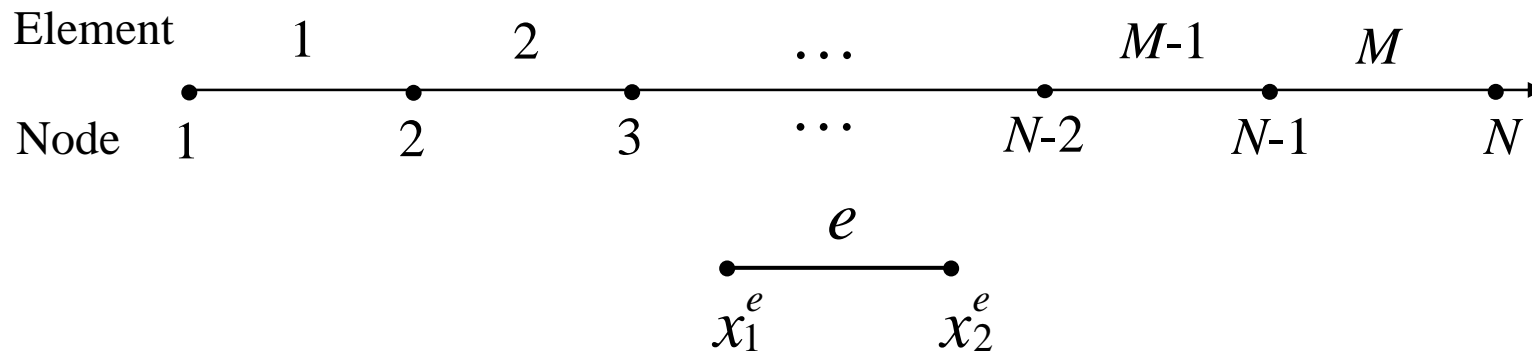
## 1D FEM (Cont'd)

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- The functional associated with this problem is

$$F(\varphi) = 0.5 \int_0^L \left[ \alpha \left( \frac{d\varphi}{dx} \right)^2 + \beta \varphi^2 \right] dx - \int_0^L f \varphi dx + \left[ \frac{\gamma}{2} \varphi^2 - q \varphi \right]_{x=L}$$

**(Prove it)!**



- We divide our computational domain into  $M$  elements with a total number of  $N$  nodes

# 1D FEM (Cont'd)

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$e \backslash j$	1	2
1	1	2
2	2	3
3	3	4
:	:	:
$M$	$M$	$M+1$

- We utilize an index table to determine the global index of each local node
- For this simple 1D case, we have  $N=M+1$  and  $x_1^e = x_e, x_2^e = x_{e+1}$

## 1D FEM (Cont'd)

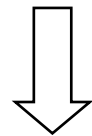
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- We approximate the unknown function  $\varphi$  by a linear approximation over each element, i.e.,

$$\varphi_e(x) = a^e + b^e x, \quad x \in \Omega_e$$

- It follows that we have

$$\varphi_1^e = a^e + b^e x_1^e, \quad \varphi_2^e = a^e + b^e x_2^e$$



Express  $a^e$  and  $b^e$  in terms of nodes values

$$\varphi^e(x) = \sum_{j=1}^2 N_j^e(x) \varphi_j^e$$

- $N_j^e(x)$  is the  $j$ th interpolation function of the  $e$ th element where  $N_1^e(x) = (x_2^e - x)/l^e$  and  $N_2^e(x) = (x - x_1^e)/l^e$  where  $l^e$  is the length of the  $e$ th element, i.e.,  $l^e = x_2^e - x_1^e$

## The Homogenous Neuman BC Case

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- For this case ( $\gamma = q = 0$ ), the functional is given by

$$F(\varphi) = 0.5 \int_0^L \left[ \alpha \left( \frac{d\varphi}{dx} \right)^2 + \beta \varphi^2 \right] dx - \int_0^L f \varphi dx$$

↓ Use elemental expansion

$$F(\varphi) = 0.5 \sum_{e=1}^M \int_{x_1^e}^{x_2^e} \left[ \alpha \left( \frac{d\varphi^e}{dx} \right)^2 + \beta (\varphi^e)^2 \right] dx - \sum_{e=1}^M \int_{x_1^e}^{x_2^e} f \varphi^e dx$$

↓  $F$  can be written as a sum of subfunctionals

$$F(\varphi) = \sum_{e=1}^M F^e(\varphi^e)$$

## The Homogenous Neuman BC Case (Cont'd)

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- The  $e$ th subfunctional is thus given by

$$F^e(\varphi^e) = 0.5 \int_{x_1^e}^{x_2^e} \left[ \alpha \left( \frac{d\varphi^e}{dx} \right)^2 + \beta (\varphi^e)^2 \right] dx - \int_{x_1^e}^{x_2^e} f \varphi^e dx$$

- It follows that we have

$$\frac{\partial F(\varphi)}{\partial \varphi_i} = \sum_{e=1}^M \frac{\partial F^e(\varphi^e)}{\partial \varphi_i}, i = 1, 2, \dots, N$$

- Notice the coefficient of any node value may be obtained through summing the associated coefficients obtained by differentiating each subfunctional

- Substituting in the  $e$ th subfunctional with the expansion

$$\varphi^e(x) = \sum_{j=1}^2 N_j^e(x) \varphi_j^e \text{ we get}$$

## The Homogenous Neuman BC Case (Cont'd)

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- It follows that we have

$$F^e(\boldsymbol{\varphi}^e) = 0.5 \int_{x_1^e}^{x_2^e} \left[ \alpha \sum_{i=1}^2 \sum_{j=1}^2 \boldsymbol{\varphi}_i^e \frac{d N_i^e}{dx} \frac{d N_j^e}{dx} \boldsymbol{\varphi}_j^e + \beta \boldsymbol{\varphi}_i^e N_i^e N_j^e \boldsymbol{\varphi}_j^e \right] dx$$

$$- \int_{x_1^e}^{x_2^e} f \sum_{i=1}^2 \boldsymbol{\varphi}_i^e N_i^e dx$$

 Differentiating w.r.t.  $\boldsymbol{\varphi}_i^e$

$$\frac{d F^e}{d \boldsymbol{\varphi}_i^e} = \sum_{j=1}^2 \boldsymbol{\varphi}_j^e \left( \int_{x_1^e}^{x_2^e} \alpha \frac{d N_i^e}{dx} \frac{d N_j^e}{dx} + \beta N_i^e N_j^e dx \right) - \int_{x_1^e}^{x_2^e} f N_i^e dx$$

- Or in matrix form  $\left\{ \frac{\partial F^e}{\partial \boldsymbol{\varphi}^e} \right\}_{2 \times 1} = [\mathbf{K}^e]_{2 \times 2} [\boldsymbol{\varphi}^e]_{2 \times 1} - [\mathbf{b}^e]_{2 \times 1}$



## The homogenous Neuman BC case (Cont'd)

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- The coefficients of the matrix  $\mathbf{K}^e$  and the vector  $\mathbf{b}^e$  are

$$K_{ij}^e = \int_{x_1^e}^{x_2^e} \left( \alpha \frac{d N_i^e}{dx} \frac{d N_j^e}{dx} + \beta N_i^e N_j^e \right) dx$$

$$b_i^e = \int_{x_1^e}^{x_2^e} f N_i^e dx$$

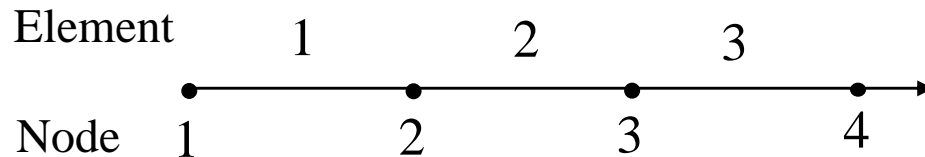
- The process of assembly involves storing the local elemental components into their proper location in the system of equations

$$\frac{\partial F}{\partial \boldsymbol{\varphi}} = \mathbf{0} \quad \Longrightarrow \quad [\mathbf{K}]_{N \times N} [\boldsymbol{\varphi}]_{N \times 1} = [\mathbf{b}]_{N \times 1}$$

# An Assembly Example

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- assuming that we have only 3 elements (4 unknowns)



- Initialization:  $\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- 1<sup>st</sup> element:  $\mathbf{K}^{(1)} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} \end{bmatrix}$ , and  $\mathbf{b}^{(1)} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \end{bmatrix}$

- Update the global system to get

## An Assembly Example (Cont'd)

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$$\mathbf{K} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ 0 \\ 0 \end{bmatrix}$$

- 2nd element:  $\mathbf{K}^{(2)} = \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} \end{bmatrix}$ , and  $\mathbf{b}^{(2)} = \begin{bmatrix} b_1^{(2)} \\ b_2^{(2)} \end{bmatrix}$

- Update the global system to get

$$\mathbf{K} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} + b_1^{(2)} \\ b_2^{(2)} \\ 0 \end{bmatrix}$$

## An Assembly Example (Cont'd)

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- By assembling the 3<sup>rd</sup> element we finally have the system of equations

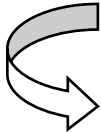
$$\mathbf{K} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} \\ 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} + b_1^{(2)} \\ b_2^{(2)} + b_1^{(3)} \\ b_2^{(3)} \end{bmatrix}$$

## The General Boundary Case

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- For the case  $q \neq 0$  or  $\gamma \neq 0$ , the functional is augmented by the subfunctional

$$F_b(\varphi) = \left[ \frac{\gamma}{2} \varphi^2 - q \varphi \right]_{x=L} \Rightarrow F_b(\varphi) = \frac{\gamma}{2} \varphi_N^2 - q \varphi_N$$



$$\frac{\partial F_b}{\partial \varphi_N} = \gamma \varphi_N - q$$

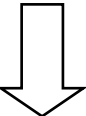
- It follows that only  $K_{NN}$  is incremented by  $\gamma$  and  $b_N$  is incremented by  $q$

# Dirichlet's Boundary Conditions

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- Dirichlet's boundary conditions are imposed by eliminating the corresponding nodal values
- Example: for the case  $N=4$ , we have

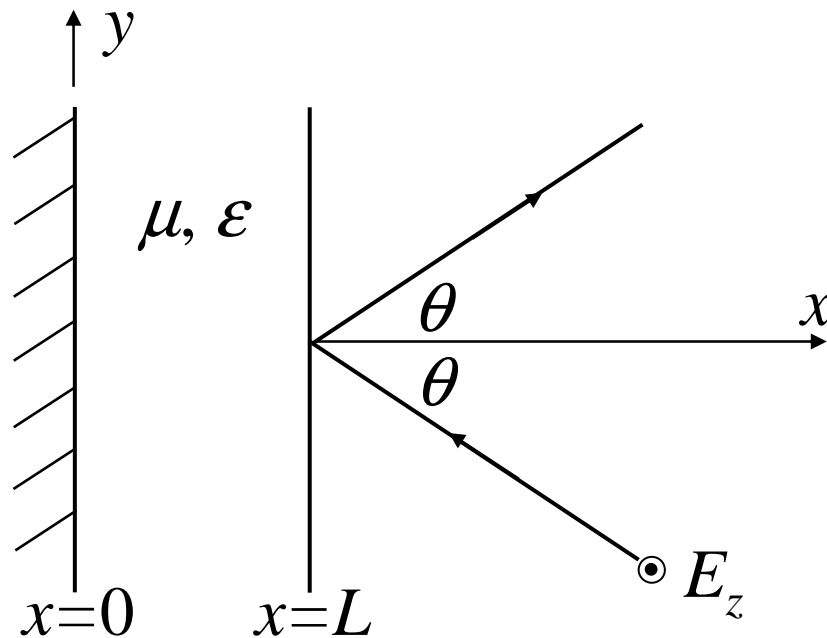
$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

 if  $\varphi_1 = p$

$$\mathbf{K} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & K_{22} & K_{23} & K_{24} \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - K_{21} p \\ b_3 - K_{31} p \\ b_4 - K_{41} p \end{bmatrix} \Rightarrow \begin{bmatrix} K_{22} & K_{23} & K_{24} \\ K_{32} & K_{33} & K_{34} \\ K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} b_2 - K_{21} p \\ b_3 - K_{31} p \\ b_4 - K_{41} p \end{bmatrix}$$

# Example

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- Determine the power reflected by this inhomogeneous metal-backed dielectric slab for a uniform incident plane wave with a  $z$  polarized electric field. Both  $\mu_r$  and  $\epsilon_r$  may vary with  $x$ .

## Example (Cont'd)

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- The expression for the incident electric field is

$$E_z^{inc}(x, y) = E_o \exp(-j\mathbf{k} \cdot \mathbf{r}), \quad \mathbf{k} = -k_o \cos \theta \mathbf{a}_x + k_o \sin \theta \mathbf{a}_y$$



$$E_z^{inc}(x, y) = E_o \exp(jk_o x \cos \theta - jk_o y \sin \theta)$$

- Notice that all field components must have a variation of  $\exp(-jk_o y \sin \theta)$  to satisfy field continuity
- The wave equation for this problem is

$$\nabla \times \left( \frac{1}{\mu} \nabla \times \mathbf{E} \right) - \omega^2 \epsilon \mathbf{E} = -j\omega \mathbf{J}$$

- Taking into account that  $\mathbf{E} = E_z \mathbf{a}_z$ ,  $\partial/\partial z = 0$  and  $\mathbf{J} = \mathbf{0}$ , we get

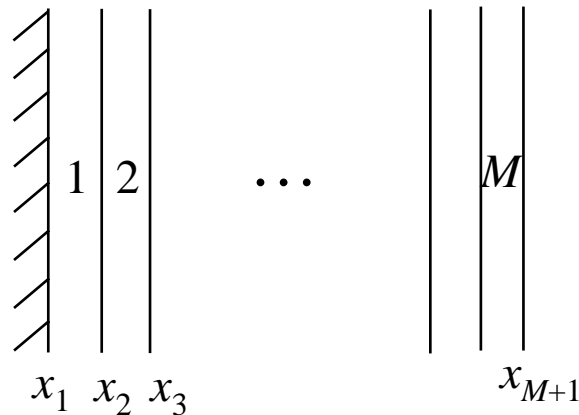
$$\left[ \frac{\partial}{\partial x} \left( \frac{1}{\mu_r} \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\mu_r} \frac{\partial}{\partial y} \right) + k_o^2 \epsilon_r \right] E_z = 0$$



## Example (Cont'd)

- Substituting for  $\frac{\partial}{\partial y} \left( \frac{1}{\mu_r} \right) = 0$ , and  $\frac{\partial E_z}{\partial y} = -j k_o \sin \theta E_z$  we get

$$\frac{\partial}{\partial x} \left( \frac{1}{\mu_r} \frac{d E_z}{dx} \right) + k_o^2 \left( \epsilon_r - \frac{\sin^2 \theta}{\mu_r} \right) E_z = 0 \text{ with the BC } E_z(0)=0$$



- An analytical solution is obtained by dividing the slab into layers  $m=1, 2, \dots, M$ , where  $\epsilon_r$  and  $\mu_r$  are assumed constant with values  $\epsilon_{rm}$  and  $\mu_{rm}$ , respectively.

## Example (Cont'd)

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- The wave equation in each layer, thus becomes

$$\frac{1}{\mu_{rm}} \frac{d^2 E_z}{d x^2} + k_o^2 \left( \epsilon_{rm} - \frac{\sin^2 \theta}{\mu_{rm}} \right) E_z = 0 \quad \Longleftrightarrow \quad \frac{d^2 E_z}{d x^2} + k_o^2 (\epsilon_{rm} \mu_{rm} - \sin^2 \theta) E_z = 0$$

which has the solution

$$E_{zm} = (A_m \exp(j k_{xm} x) + B_m \exp(-j k_{xm} x)) \exp(-j k_o \sin \theta)$$

$$k_{xm} = k_o \sqrt{\epsilon_{rm} \mu_{rm} - \sin^2 \theta}$$

- The analytical solution is obtained by enforcing continuity of the electric and magnetic field components at the layers interface to get

## Example (Cont'd)

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$$R_{m+1} = \frac{\eta_{m+1,m} + R_m \exp(-2j k_{xm} x_{m+1})}{1 + \eta_{m+1,m} R_m \exp(-2j k_{xm} x_{m+1})} \exp(2j k_{xm+1} x_{m+1})$$

$$R_m = \frac{B_m}{A_m}, \quad \eta_{m+1,m} = \frac{\mu_{rm} k_{xm+1} - \mu_{rm+1} k_{xm}}{\mu_{rm} k_{xm+1} + \mu_{rm+1} k_{xm}}$$

$$R_1 = \frac{B_1}{A_1} = -1, \quad (\text{conductor}) \quad \textbf{(Prove it)!}$$

## FEM Solution

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- Our problem is given by

$$\frac{d}{dx} \left( \frac{1}{\mu_r} \frac{d E_z}{dx} \right) + k_o^2 \left( \epsilon_r - \frac{\sin^2 \theta}{\mu_r} \right) E_z = 0 \text{ with the BC } E_z(0)=0$$

- A boundary condition at  $x=L$  to have a finite computational domain
- For  $L \leq x$ , we have

$$E_z(x, y) = (E_o \exp(jk_o x \cos \theta) + R E_o \exp(-jk_o x \cos \theta)) \exp(-jk_o y \sin \theta)$$



$$E_z(x, y) = E_z(x) \exp(-jk_o y \sin \theta)$$

- Differentiating  $E_z(x)$  relative to  $x$  we get


$$\frac{d E_z(x)}{dx} = j k_o \cos \theta (E_o \exp(j k_o x \cos \theta) - R E_o \exp(-j k_o x \cos \theta))$$

## FEM Solution (Cont'd)

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- Manipulating we get

$$\frac{dE_z(x)}{dx} = 2jk_o \cos\theta E_o \exp(jk_o x \cos\theta) - jk_o \cos\theta E_z(x)$$


$$\left( \frac{dE_z(x)}{dx} + jk_o \cos\theta E_z(x) \right) \Big|_{x=L^+} = 2jk_o \cos\theta E_o \exp(jk_o L \cos\theta)$$

- Utilizing the continuity of the electric and magnetic field we may convert this boundary condition into

$$\left( \frac{1}{\mu_r} \frac{dE_z(x)}{dx} + jk_o \cos\theta E_z(x) \right) \Big|_{x=L^-} = 2jk_o \cos\theta E_o \exp(jk_o L \cos\theta)$$

## FEM Solution (Cont'd)

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- It follows that our problem is given by

$$\frac{d}{dx} \left( \frac{1}{\mu_r} \frac{d E_z}{dx} \right) + k_o^2 \left( \epsilon_r - \frac{\sin^2 \theta}{\mu_r} \right) E_z = 0 \quad \text{with } E_z(0)=0$$

$$\left( \frac{1}{\mu_r} \frac{d E_z(x)}{dx} + j k_o \cos \theta E_z(x) \right) \Big|_{x=L} = 2 j k_o \cos \theta E_o \exp(j k_o L \cos \theta)$$

- Comparing with our 1D FEM formulation we have

$$\varphi = E_z(x), \quad \alpha = 1/\mu_r, \quad \beta = -k_o^2 \left( \epsilon_r - \frac{\sin^2 \theta}{\mu_r} \right)$$

$$\gamma = j k_o \cos \theta, \quad q = 2 j E_o k_o \cos \theta \exp(k_o L \cos \theta)$$

- Once the field is solved, the reflection coefficient is given by

$$R = \frac{E_z(x) - (E_o \exp(j k_o L \cos \theta))}{E_o \exp(-j k_o L \cos \theta)}$$