Analysis of Dynamic Circuits

Circuits with capacitors and inductors

Current options:

Circuit type \ Information needed	Complete response to step, sinusoid, or exponential	Steady-state response to sinusoid.
Linear, 1^{st} or 2^{nd} order	Diff. eqns (Hard) 1	Phasors 2
Linear, many L's + C's	Diff eqns (Harder) 3	Phasors 4
Non-linear	Diff eqns (Really hard) 5	? 6

Nice things about phasor analysis: turns diff. eqns into algebraic eqns.

Not so nice things: only applies to steady-state sinusoidal response for linear circuits

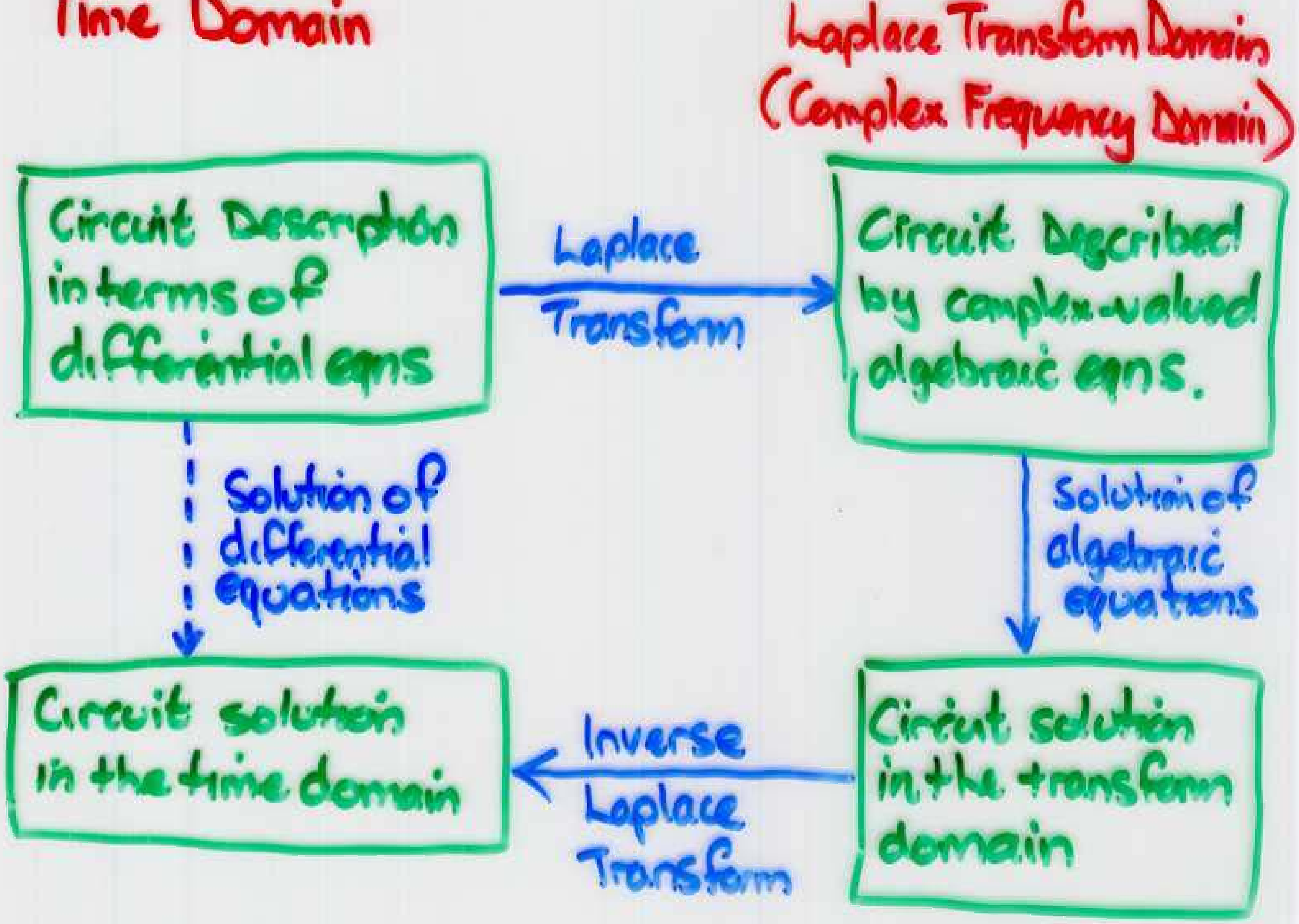
What would we like: to get complete response (transient + steady-state) for any linear circuit using simple algebra

Would be even better if Phasor analysis was a special case of this new method.

* Fortunately such a tool does exist.
* It is the Laplace Transform
* Unfortunately it only helps the analysis of linear circuits

* Laplace transforms are not only important for circuit analysis + design, but they form the mathematical foundation of linear control system design, with applications in robotics, Canada Arm, active flight control, chemical process control, etc

Transform Domain Analysis



A similar diagram applies for steady-state analysis of circuits excited by sinusoidal signals. The "transform" in that case is the phasor.

The advantage of the Laplace Transform is that it applies to essentially all useful signals + provides both steady-state + transient responses.

LAPLACE TRANSFORMS

* We will only look at the "unilateral" Laplace transform
* This is designed for signals which are zero for $t<0$
* Extensions to more general transforms will be considered next year (or perhaps in the Fall)

* Definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

* Notes

$s = \sigma + j\omega$ is a complex number

$\Rightarrow$ $F(s)$ is a complex number (usually expressed in terms of magnitude and phase)

* For simplicity, we will restrict are attention to $f(t)$'s which are continuous for $t>0$. However we will allow a discontinuity at $t=0$
* We will capture the consequences of the circuit behaviour for $t<0$ in the initial conditions, &

$$e^{-(\sigma+j\omega)t} = e^{-\sigma t}\, e^{-j\omega t}$$

$$= e^{-\sigma t}\left[\cos\omega t - j\sin\omega t\right].$$

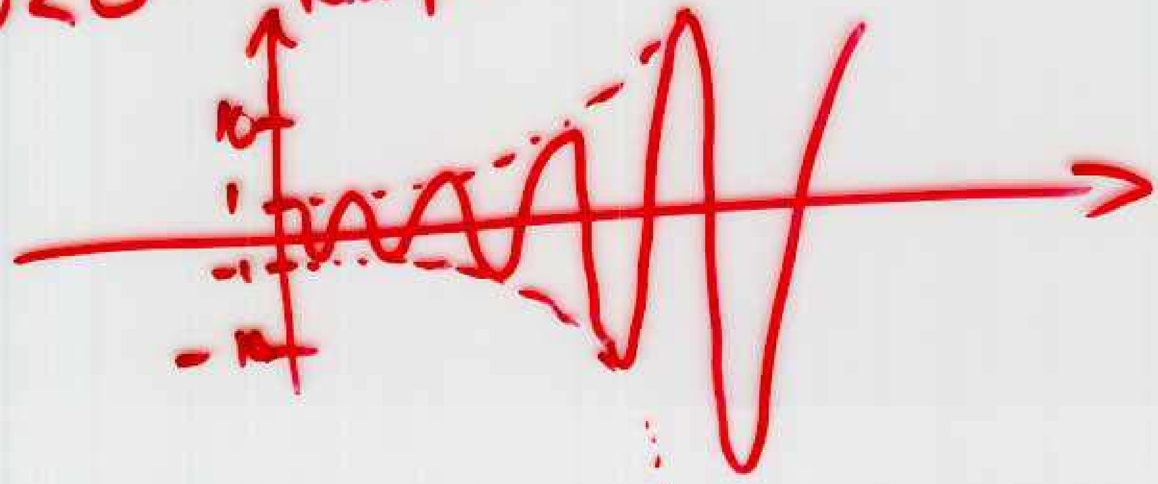
Real part.

$\sigma > 0$

Imag part

$\sigma < 0$

Real part



To define a function $f(t)$ which is zero for $t<0$ and something else for $t\geq 0$, we can write

$$f(t)=\begin{cases} 0 & t<0 \\ g(t) & t\geq 0 \end{cases}$$

However, this can be quite clumsy.

Define

$$u(t)=\begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$$

This is the unit step function.

Then $$f(t)=g(t)\,u(t).$$

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

When does this integral exist?

ie when is

$$\left| \int_0^\infty f(t) e^{-st} dt \right| < \infty \quad ?$$

well

$$LHS = \left| \int_0^\infty f(t) e^{-st} dt \right| \leqslant \int_0^\infty \left| f(t) e^{-st} \right| dt$$

$$\leqslant \int_0^\infty |f(t)| e^{-\sigma t} dt.$$

Now assume that there is an M and α such that.

$$|f(t)| < M e^{\alpha t} \quad \text{for all} \quad t > 0$$

$$\Rightarrow LHS \leqslant M \int_0^\infty e^{(\alpha - \sigma) t} dt.$$

$$= M \left. \frac{e^{(\alpha - \sigma) t}}{(\alpha - \sigma)} \right]_0^\infty$$

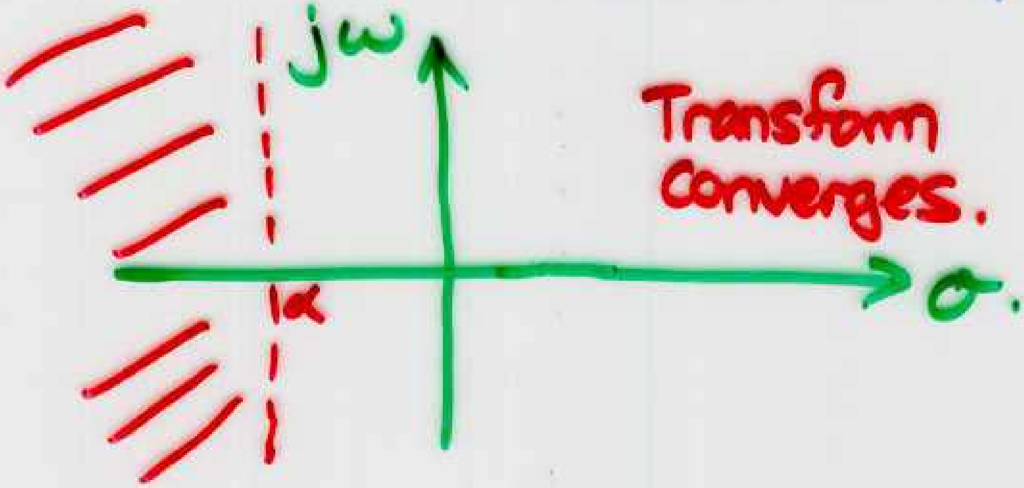
To ensure that the upper limit is finite, we must have $\alpha - \sigma < 0$

$\Rightarrow$ integral will converge if $\sigma > \alpha$.

Therefore, given $f(t)$, M and α

$$F(\sigma + j\omega) \text{ exists for all } \sigma > \alpha$$

That is, the transform exists on a "half-plane"



Example: $f(t) = e^{-at} u(t)$, $a > 0$

$$F(s) = \int_0^\infty e^{-at} e^{-st} dt$$

$$= \frac{1}{s+a}, \text{ for } \sigma > -a$$

What does $e^{-at} u(t)$ look like?

In fact, as a gets small,
$f(t)$ starts to look like $u(t)$.

Hence $U(s) = \frac{1}{s}$

Homework: Show that this is true by direct application of the formula

* I said earlier that Laplace transforms simplify the solution of differential equations.
* To show that, we need to find

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt$$

Now integrate by parts

$$= s\int_0^\infty f(t)e^{-st}\,dt + f(t)e^{-st}\Big]_0^\infty$$

$$\Rightarrow \mathcal{L}\{f'(t)\} = sF(s) - f(0^+)$$

Note here the 0^+ because we have allowed a discontinuity at $t=0$.

* Linearity

A fundamental property of the Laplace transform is that it is linear

i.e. $\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$

without this property it would not be much good to us.

Example.

What is $\mathcal{L}\{\cos(\omega_0 t)\,u(t)\}$?

Well $\cos \omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right].$

$\Rightarrow$, by linearity,

$$\mathcal{L}\{\cos(\omega_0 t)\,u(t)\} = \frac{1}{2}\mathcal{L}\{e^{j\omega_0 t}u(t)\} + \frac{1}{2}\mathcal{L}\{e^{-j\omega_0 t}u(t)\}$$

But we already know that

$$\mathcal{L}\{e^{at}u(t)\} = \frac{1}{s+a}.$$

Therefore.

$$\mathcal{L}\{\cos(\omega_0 t)\,u(t)\} = \frac{1}{2}\left[\frac{1}{s+j\omega_0} + \frac{1}{s-j\omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s-j\omega_0 + s+j\omega_0}{s^2+\omega_0^2}\right]$$

$$= \frac{s}{s^2+\omega_0^2}$$

* Uniqueness

For this "unilateral" Laplace transform, the mapping is one-to-one.

* That means if $F_1(s) = F_2(s)$ then $f_1(t) = f_2(t)$
* There are other transforms for which this does not hold.

* Inversion

Given an $F(s)$, we can find the corresponding $f(t)$ using.

$$f(t) = \frac{1}{2\pi j} \int_{\beta - j\infty}^{\beta + j\infty} F(s) e^{st} \, ds.$$

where β ~~&~~ lies in the convergence region ie $\beta > \alpha$ in previous notation.

This is an integral with respect to a complex number, and although it can be simplified, we'd like to avoid it.

* This integral can often be avoided by tabulating a small number of "transform pairs" $f(t), F(s)$ and then using the properties of the transform to determine other pairs. More Next Thursday!

Time Delay

If $\mathcal{L}\{f(t)u(t)\} = F(s)$

What is the Laplace Transform of a shifted version of $f(t)$?

$$\mathcal{L}\{f(t-\tau)u(t-\tau)\} = \int_0^\infty f(t-\tau)u(t-\tau)e^{-st}\,dt$$

$$= \int_\tau^\infty f(t-\tau)\,e^{-st}\,dt$$

let $x = t-\tau$

$$= \int_0^\infty f(x)\,e^{-s(\tau+x)}\,dx.$$

$$= e^{-s\tau}\int_0^\infty f(x)e^{-sx}\,dx$$

$$= e^{-s\tau}F(s).$$

$\Rightarrow$ We do not need to calculate the Laplace Transform again!

* Frequency Shift Property.

If $\mathcal{L}\{f(t)u(t)\} = F(s)$,

what is $\mathcal{L}\{e^{-at} f(t)u(t)\}$?

$$\mathcal{L}\{e^{-at} f(t)u(t)\} = \int_0^\infty e^{-at} f(t) e^{-st} dt$$

$$= \int_0^\infty f(t) e^{-(s+a)t} dt$$

$$= F(s+a).$$

* Time Multiplication property.

If $\mathcal{L}\{f(t)u(t)\} = F(s)$, what is $\mathcal{L}\{t f(t) u(t)\}$?

This is useful for critically damped circuits

Look at $\frac{d}{ds} F(s) = \int_0^\infty \frac{d}{ds} f(t) e^{-st} dt.$

$$= \int_0^\infty f(t) \frac{d}{ds} e^{-st} dt.$$

$$= -\int_0^\infty t f(t) e^{-st} dt$$

Important note:

We will not discuss the impulse function nor the convolution theorem in this course.

$$\Rightarrow \mathcal{L}\{t f(t) u(t)\} = -\frac{dF(s)}{ds}$$

Table of Laplace Transform Pair's

Using our examples + the properties we can construct the following table.

$f(t),\ t \geq 0$	$F(s)$
$u(t)$	$\frac{1}{s}$
e^{-at}	$1/(s+a)$
$\sin \omega_0 t$	$\omega_0/(s^2+\omega_0^2)$
$\cos \omega_0 t$	$s/(s^2+\omega_0^2)$
$e^{-at} f(t)$	$F(s+a)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{d^k f(t)}{dt^k}$	$s^k F(s) - s^{k-1} f(0^+) - s^{k-2} f'(0^+) - \cdots - f^{(k-1)}(0^+)$
te^{-at}	$\frac{1}{(s+a)^2}$
$e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$
$e^{-at} \cos \omega_0 t$	$\frac{(s+a)}{(s+a)^2+\omega_0^2}$

Inversion of the Laplace Transform.

$$f(t) = \frac{1}{2\pi j}\int_{\beta - j\infty}^{\beta + j\infty} F(s)e^{st}\,ds$$

But this is time consuming to do all the time

Fortunately, most of the $F(s)$'s we deal with are ratios of polynomials:

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Can we make our job easier?

Facts.

1. Any polynomial with real coefficients can be factorized into a product of first and second-order factors
2. We can use this fact to write $F(s)$ as a sum of simpler terms.
3. We know how to inverse transform the simpler terms.
4. By linearity, we can invert each term to get the inverse of the whole.

Example

$$F(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2}.$$

What are A, B?

General procedure involves residues, but for isolated 1st order, real roots, just reconstruct the ratio!

$$\frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

$$= \frac{(A+B)s + 2A+B}{(s+1)\ (s+2)}$$

$$\Rightarrow \left.\begin{array}{l} A+B = 1 \\ 2A+B = 3 \end{array}\right\} \Rightarrow A=2,\ B=-1$$

Using the tables, $\frac{2}{s+1}$ inverts to $2e^{-t}$, $t \geq 0$

$\frac{-1}{s+2}$ inverts to $-e^{-2t}$, $t \geq 0$

$$\Rightarrow f(t) = 2e^{-t} - e^{-2t}, \quad t \geq 0$$

Special cases.

Repeated roots

if we have a factor $(s-r)^m$ in the denominator the corresponding partial fraction is

$$\frac{b_1}{s-r} + \frac{b_2}{(s-r)^2} + \cdots + \frac{b_m}{(s-r)^m}$$

Second-order factors

* When taking partial fractions we often encounter second order factors.
* We can deal with these in two ways.
  (i) linear factors with complex roots, hence complex multipliers
  (ii) as second-order factors directly.

* Example of (ii)

$$F(s) = \frac{N(s)}{(s+a+j\omega)(s+a-j\omega)} = \frac{N(s)}{(s+a)^2+\omega^2}$$

if $N(s) = s+a$, $\quad f(t) = e^{-at}\cos\omega t$

if $N(s) = d$ $\quad f(t) = \frac{d}{\omega} e^{-at}\sin\omega t$

* Further example (prove for homework)

$$F(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\Rightarrow f(t) = \frac{\omega_n}{b} e^{-\zeta\omega_n t} \sin(\omega_n b t) \quad , t \geq 0$$

where $\zeta < 1$ and $b = \sqrt{1-\zeta^2}$

What happens when $\zeta = 1$, $\zeta > 1$?

* Example of approach (II). (prove for homework)

After taking the linear factors + expanding

$$F(s) = \frac{c+jd}{s+a-j\omega} + \frac{c-jd}{s+a+j\omega}$$

$$= \frac{2(cs + ca - \omega d)}{(s+a)^2 + \omega^2}$$

$$\Rightarrow f(t) = 2me^{-at} \cos(\omega t + \phi) \quad , t \geq 0$$

$$m = \sqrt{c^2+d^2} \quad , \phi = \tan^{-1}\left(\frac{d}{c}\right)$$