- Note that since $X(s) \to \infty$ as $s \to$ a pole, the ROC cannot contain any poles
- We will see that the poles define the boundaries of the ROC.
- The ROC is required in order to invert the Laplace Transform

**EXAMPLE 6**

$$x(t) = e^{at}u(t),$$

$$\Rightarrow \quad X(s) \quad \int_{-\infty}^{\infty} e^{at}u(t)e^{-st}\,dt$$

$$\int_{0}^{\infty} e^{-(s-a)t}\,dt$$

$$\frac{1}{s-a} \; e^{(s-a)t}\Big]_0^{\infty}$$

$$\Rightarrow \quad X\,s) = \frac{}{s-a} \; e^{-(\sigma-a)t} e^{-j\omega t}\Big]_0^{\infty}$$

if $\sigma > a$ $\quad e^{(\sigma-a)t} \to 0$ as $t \to \infty$

$$\Rightarrow \quad X(s) \quad \frac{1}{s-a}, \quad \mathrm{Re}\, s) > a$$

The Laplace Transform does not exist f $\sigma < a$

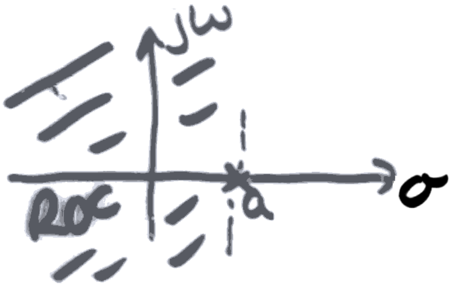
**EXAMPLE 6.2**

$$y(t) = -e^{-at}u(-t)$$

$$Y(s) = \int_{-\infty}^{\infty} e^{-at}u(-t)\, e^{st}\, dt$$

$$\int_{-\infty}^{0} e^{(\sigma-a)t} e^{-j\omega t}\, dt$$

$$-\ \frac{1}{s-a}\, e^{-(\sigma-a)t} e^{-j\omega t} \Big]_{-\infty}^{0}$$




$$-\ \frac{1}{s-a} \qquad \mathrm{Re}\{s\} < a$$

Hence ~~fi~~ both $e^{at}u(t)$ and $-e^{-at}u(-t)$ have the same Laplace Transform, but different ROCs

Hence we need to know the ROC to invert the Laplace Transform

# NOTES.

In 2CJ4 we looked at the unilateral Laplace Transform

$$x(t) \overset{L_u}{\longleftrightarrow} X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

As I mentioned at that time, this year we will look at the bilateral case

$$x(t) \overset{L}{\longleftrightarrow} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Why?

allows us to deal with non-causal systems and signals which start before $t=0$

Consequences

Algebra + Properties very similar

However we need ROC + t.s properties

Need ROC to do inversion

# BILATERAL LAPLACE TRANSFORM

$$x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$

- Transform is not unique, and hence not invertible, unless you specify the Region of Convergence (ROC) i.e., the set of values of $s$ for which integral converges.
- We now develop some properties of the ROC which make it easier to determine

$$X(s) = \frac{N(s)}{D(s)}$$

$$s = \sigma + j\omega.$$

## PROPERTIES OF ROC

① Does not contain any poles.

② Since convergence of integral implies that

$$\boxed{I(\sigma) = \int |x(t)|\, e^{-\sigma t} dt < \infty}$$

The ROC depends only on $\sigma$. Hence it consists of a vertical strip in the s-plane

③ ROC for a finite duration signal:

$$I(\sigma) = \int |x(t)| e^{-\sigma t} dt$$

Suppose $x(t)$ is zero outside $t \in [a, b]$ and that $|x(t)| \leq M$

Then $I(\sigma) \leq \int_a^b M e^{-\sigma t} dt.$

$$\begin{cases} \left.\frac{-A}{\sigma} e^{-\sigma t}\right]_a^b & , \sigma \neq 0 \\ A(b-a) & \sigma = 0 \end{cases}$$

This is finite for all finite values of $\sigma$

$\Rightarrow$ ROC of a finite duration signal is the whole s-plane

④ Infinitely long signals of "exponential order"
(almost all practical signals are of exponential order)

Assume $\quad x(t) \leq A e^{\sigma_p t}, t > 0$

$\qquad\qquad x(t) < A e^{\sigma_n t}, \ t < 0$

These bounds get tighter as $A, \sigma_p$ get smaller, $\sigma_n$ gets larger

What does $I(\sigma)$ look like in these cases?

$$I(\sigma) = I_+(\sigma) + I_-(\sigma)$$
$$= \int_0^{\infty} |x(t)| e^{-\sigma t} dt + \int_{-\infty}^{0} |x(t)| e^{-\sigma t} dt$$

$$I_-(\sigma) = \int_{-\infty}^{0} |x(t)| e^{-\sigma t} dt$$
$$\leq A \int_{-\infty}^{0} e^{(\sigma_n - \sigma)t} dt$$
$$= \frac{A}{\sigma_n - \sigma} e^{(\sigma_n - \sigma)t} \Big]_{-\infty}^{0}$$

Similarly,

$$I_+(\sigma) = \int_{0}^{\infty} |x(t)| e^{-\sigma t} dt$$
$$\leq \frac{A}{\sigma_p - \sigma} e^{(\sigma_p - \sigma)t} \Big]_{0}^{\infty}$$

Hence $I_-(\sigma)$ is finite for $\sigma < \sigma_n$.

$I_+(\sigma)$ is finite for $\sigma > \sigma_p$

$\Rightarrow$ $I(\sigma)$ is finite for $\sigma_p < \sigma < \sigma_n$

(Note that this region maybe empty!)

This allows us to make following conclusions

Left-sided signal, $x(t) = 0$ for $t > b$ $\Rightarrow$ ROC is $\sigma < \sigma_n$.

Right-sided signal, $x(t) = 0$ for $t < a$ $\Rightarrow$ ROC is $\sigma > \sigma_p$

Two-sided signal $\Rightarrow$ ROC is $\sigma_p < \sigma < \sigma_n$

Fig 6.12

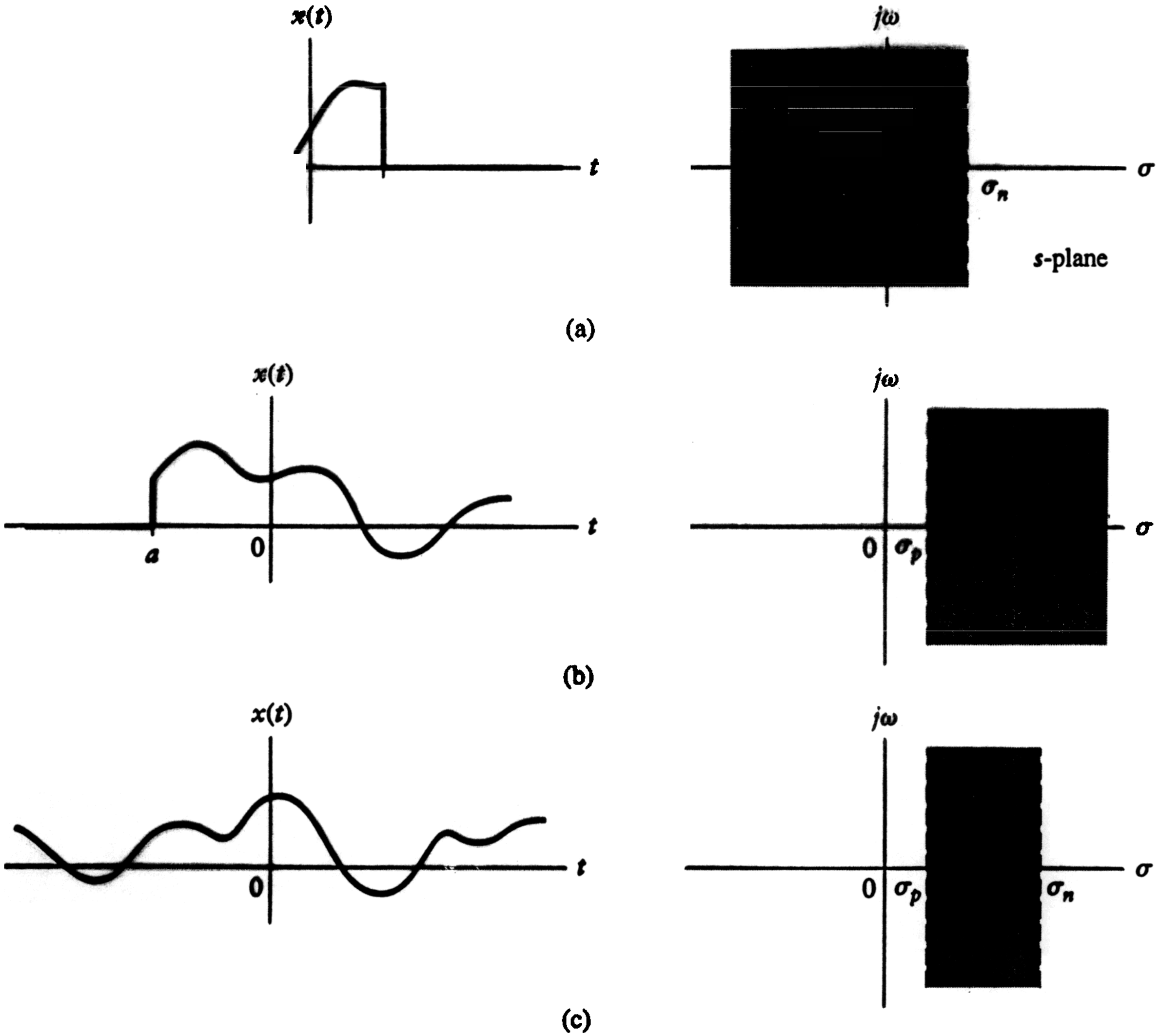



**FIGURE 6.12** Relationship between the time extent of a signal and the ROC. (a) A left-sided signal has ROC to the left of a vertical line in the $s$-plane. (b) A right-sided signal has ROC to the right of a vertical line in the $s$-plane. (c) A two-sided signal has ROC given by a vertical strip in the $s$-plane of finite width.

EXAMPLE 6.12

Identify the ROC of

$$x_1(t) = e^{-2t}u(t) + e^{-t}u(-t)$$

$$x_2(t) = e^{-t}u(t) + e^{-2t}u(-t)$$

$$I_1(\sigma) = \int |x_1(t)| e^{-\sigma t} dt$$

$$= \int_{-\infty}^{0} e^{-(1+\sigma)t} dt + \int_{0}^{\infty} e^{-(2+\sigma)t} dt$$

$$= \underbrace{\frac{-1}{1+\sigma} e^{-(1+\sigma)t} \Big]_{-\infty}^{0}}_{\text{finite for } \sigma < -1} + \underbrace{\frac{-1}{2+\sigma} e^{-(2+\sigma)t} \Big]_{0}^{\infty}}_{\text{finite for } \sigma > -2}$$

$\Rightarrow$ ROC is $-2 < \sigma < -1$

Note that

$$X_1(s) = \frac{-1}{s+1} + \frac{1}{s+2} = \frac{-1}{(s+1)(s+2)}$$

$\Rightarrow$ $X_1(s)$ has poles at $s=-1$ and $s=-2$ and these form the boundary of the RO

For $x_2(t)$,

$$I_2(\sigma) = \int_{-\infty}^{0} e^{-(2+\sigma)t}\,dt + \int_{0}^{\infty} e^{-(1+\sigma)t}\,dt$$

$$= \underbrace{\frac{-1}{2+\sigma} e^{-(2+\sigma)t}\Big]_{-\infty}^{0}}_{\text{finite for } \sigma<-2} + \underbrace{\frac{-1}{1+\sigma} e^{-(1+\sigma)t}\Big]_{0}^{\infty}}_{\text{finite for } \sigma>-1}$$

$\Rightarrow$ ROC is $\sigma<-2 \cap \sigma>-1$

$\Rightarrow$ There is no ROC.

$\Rightarrow$ $X_2(s)$ does not exist anywhere.

However

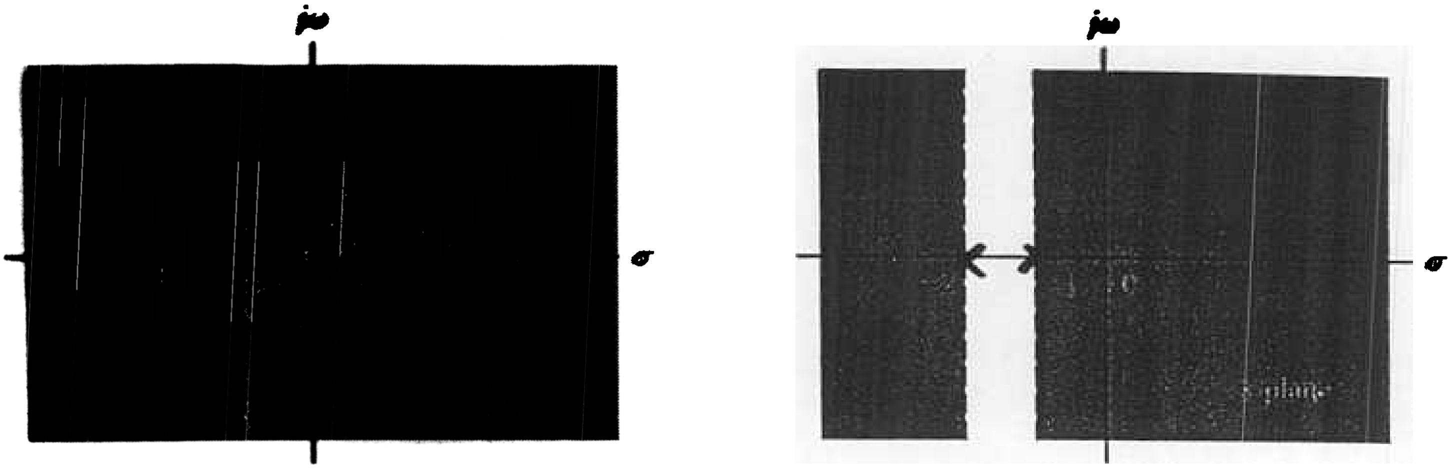
$$x_2(t) = e^{-t}u(t) + e^{-2t}u(-t)$$

and.

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+}, \quad Re\{s\} >$$

$$e^{-2t}u(-t) \leftrightarrow \frac{-}{s+2}, \quad Re\{s\} < -2.$$

# Since ROC's are disjoint, Laplace transform of sum is not defined




(a) (b)

**FIGURE 6.13** ROCs for signals in Example 6.12. (a) The shaded regions denote the ROCs of each individual term, $e^{-2t}u(t)$ and $e^{-t}u(-t)$. The doubly shaded region is the intersection of the individual ROCs and represents the ROC of the sum. (b) The shaded regions represent the individual ROCs of $e^{-2t}u(-t)$ and $e^{-t}u(t)$. In this case there is no intersection and the Laplace transform of the sum does not converge for any value of $s$.