Root Locus Analysis

We have just seen three examples.

- The root locus is the set of points which describe the closed loop poles of

$$T(s) = \frac{G(s)H(s)}{1+G(s)H(s)}$$

as a parameter of $H(s)$ varies

- Typically, $H(s) = K\tilde{H}(s)$ and K ranges from 0 to ∞
- We will consider that case only
- Where are the closed loop poles?

Let $G(s)H(s) = KG(s)\tilde{H}(s) = \dfrac{K\tilde{P}(s)}{s^k \tilde{Q}_1(s)} = \dfrac{KP(s)}{Q(s)}$

Here, $\tilde{P}, \tilde{Q}_1$, P and Q are polynomials

- Hence. $T(s) = \dfrac{G(s)H(s)}{1+G(s)H(s)} = \dfrac{KP(s)}{Q(s)+KP(s)}$

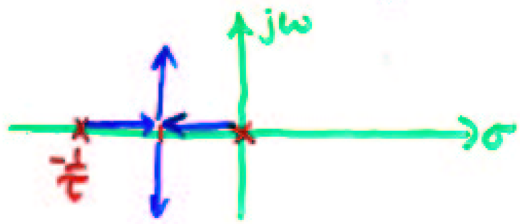
$\Rightarrow$ closed loop poles are roots of

$$A(s) = KP(s) + Q(s)$$

- Therefore root locus shows us how the closed loop poles move as K is varied.

- Example 1: $G(s) = \frac{1}{s+a}$, $\bar{H}(s) = 1$

- Example 2: $G(s) = \frac{1}{s(s\tau+1)}$, $\bar{H}(s) = 1$



- These examples are quite straight forward, because we can find a formula for the closed loop poles as a function of K ($A(s)$ is linear or quadratic)

- Example 3: Fig 9.26.

$$G(s)\bar{H}(s) = \frac{6}{(s+1)(s+2)(s+3)}$$

- As seen from last example, stability of a system may change with K.

- Hence important to be able to find K so that system is on verge of being unstable
  i.e. has poles on $j\omega$-axis

- How to do this?

  1. Use Routh-Hurwitz analysis, but we will not cover that in this course (Sect 9.12)

  2. Simply recognize that if system has poles on $j\omega_c$-axis, then $A(s)$ can be factorized as

poles $A(s)=0$

$$A(s) = (s^2 + \omega_c^2)\,\tilde{A}(s) \qquad (*)$$

where ω_c is not yet known.
If we can find a ω_c and K such that this factorization exists, then we are done

Basics:

i) Use polynomial division to write
$$A(s) = (s^2 + \omega_c^2)\,\tilde{A}(s) + R(s)$$
$R(s)$ is "remainder" / residual

ii) Factorization in (*) exists if $R(s) \equiv 0, \forall s$
Solve this equation for K and ω_c

SIDEBAR ON POLYNOMIAL DIVISION

Given $A(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$
and $B(s) = b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0$

with $m < n$, find C(s) and D(s) such that

$$A(s) = B(s)\,C(s) + D(s)$$

Procedure.

$$
\begin{array}{r|l}
 & \frac{a_n}{b_m} s^{n-m} + \dots \\
\hline
b_m s^m + \dots + b_0 & a_n s^n + \dots \quad a_1 s + a_0 \\
 & -(a_n s^n + \dots \qquad\qquad) \qquad \frac{a_n}{b_m} s^{n-m} \times B(s) \\
\hline
 & 0 + \gamma_{n-1} s^{n-1} + \gamma_{n-2} s^{n-2} + \dots + \gamma_0 \qquad \text{subtract}
\end{array}
$$

Continue until order of residual is $< m$

Then C(s) is on top
D(s) is on bottom (residual)

We will only encounter simple examples

but numerical techniques exist for more general cases (3SK4 ?)

- Recall that the root locus is the set of paths of the roots of.

$$A(s) = K\,P(s) + Q(s) = 0 \quad \text{as} \quad K \text{ goes from } 0 \to \infty$$

- These days root loci are usually computed numerically (3SK4) but we will now state some properties which allow us to sketch the loci + pland design strategies

1. Root locus starts from poles of $G(s)\tilde{H}(s)$.

   i) when $K=0$, $A(s) = Q(s)$

   ii) $G(s)\tilde{H}(s) = \dfrac{P(s)}{Q(s)}$ $\Rightarrow$ start from poles of $G(s)\tilde{H}(s)$

2. Root locus ends at zeros of $G(s)\tilde{H}(s)$, including those at ∞.

   i) $A(s)=0$ can be rewritten as.

$$\frac{P(s)}{Q(s)} = \frac{-1}{K}$$

   $\Rightarrow$ as K gets large, roots of $A(s)=0$ move towards zeros of $G(s)\tilde{H}(s)$.

3. Root locus is symmetric with respect to real axis.

   For real-valued systems poles are either real or come in complex conjugate pairs

Let M be the order of $P(s)$
N be the order of $Q(s)$ $\Rightarrow G(s)\tilde{H}(s) = \dfrac{\prod_{k=1}^{M}(s-c_k)}{\prod_{k=1}^{N}(s-d_k)}$

4. As K gets large.

i) $N-M$ branches head off to $s=\infty$

ii) They do so at angles $\theta_k = \dfrac{(2k+1)\pi}{N-M}$

$k = 0, 1, \ldots, N-M-1$

iii) The resulting asymptotes intersect at the point

$$\sigma_0 = \frac{\sum_k d_k - \sum_k c_k}{N-M}$$

$$= \frac{\begin{pmatrix}\text{sum of}\\ \text{finite poles}\end{pmatrix} - \begin{pmatrix}\text{sum of}\\ \text{finite zeros}\end{pmatrix}}{N-M}$$

5. The value of K for which the root locus has poles in a particular place can be found by polynomial division.

6. ~~The value~~ If $s = s_x$ denotes a value for s where loci intersect, then.

$$\frac{d}{ds}\left(\frac{1}{G(s)H(s)}\right) = 0.$$

7. Magnitude criterion: If $s = s_\ell$ is on the root locus, then the corresponding value of gain is

$$K = \frac{|Q(s_\ell)|}{|P(s_\ell)|}$$

$$= \frac{\text{prod. distances from } s_\ell \text{ to poles of } G(s)H(s)}{\text{prod. distances from } s_\ell \text{ to zeros of } G(s)H(s)}$$