EXAMPLE 9.10.

Sketch the root locus of $K\,G(s)\tilde{H}(s)$ where

$$G(s)\tilde{H}(s) = \frac{6}{(s+1)(s+2)(s+3)}$$

STEP 1: plot starting points, poles of $G(s)\tilde{H}(s)$

$= -1, -2, -3$

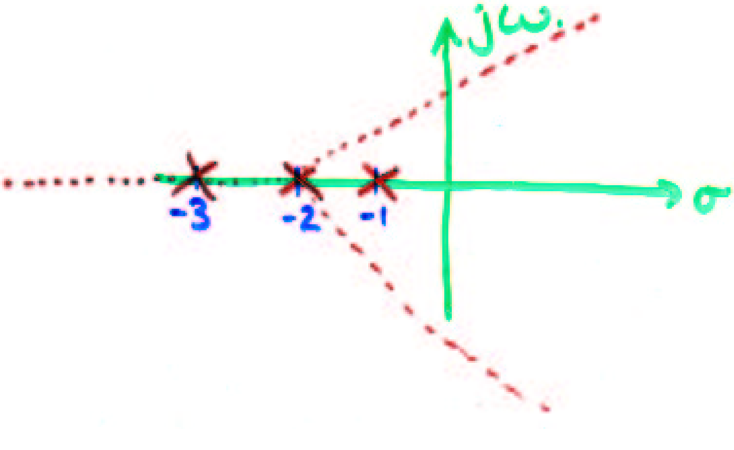
STEP 2: Find end points: $G(s)\tilde{H}(s)$ only has zeros at ∞ $(M=0)$

STEP 3: FIND asymptotes: $N-M=3$, so there are three paths that head off to ∞.

- they do so at angles of 60°, 180° and 300°
- The asymptotes intersect at

$$\sigma_0 = \frac{-1-2-3}{3} = -2.$$

PROGRESS UP TILL NOW.



STEP 4: where do loci intersect

$$\frac{1}{G(s)H(s)} = \frac{(s+1)(s+2)(s+3)}{6}$$

$$\Rightarrow \frac{d}{ds}\left(\frac{1}{G(s)H(s)}\right) = 3s^2 + 12s + 1$$

Potential intersection points are where

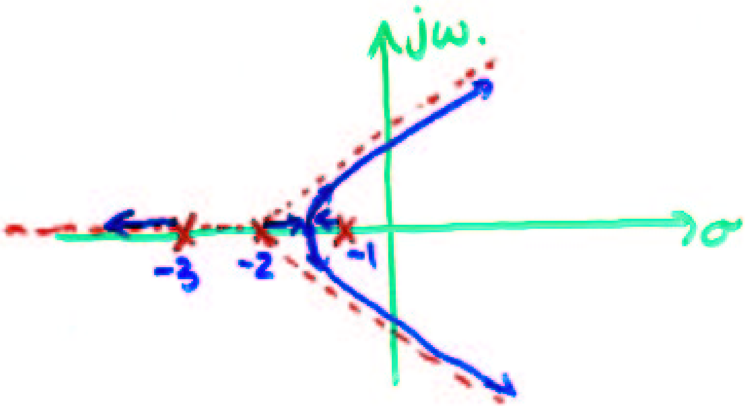
$$3s^2 + 12s + 1 = 0$$

$$\Rightarrow s \approx -1.423, -2.577$$

If they intersect at -2.577, then they cannot follow asymptotes $\Rightarrow$ intersect at $s = -1.423$

The value of the gain is given by the magnitude criterion

$$K_{intersect} = |1-1.423| \times |2-1.423| \times |3-1.423|$$
$$= 0.385$$



- All that remains is to find K such that poles are on $j\omega$-axis.

$$A(s) = s^3 + 6s^2 + 11(s) + 6(k+1)$$
$$= K\,P(s) + Q(s)$$

Divide by $(s^2+\omega_c^2)$, ω_c^2 not yet known.

$$\begin{array}{r|l}
 & s + 6 \\
\hline
s^2+\omega_c^2 & s^3+6s^2+11s+6(k+1) \\
 & -\ s^3+0+\omega_c^2 s+0 \\
\hline
 & 6s^2+(11-\omega_c^2)s+6(k+1) \\
 & -\ 6s^2+0 \quad +6\omega_c^2 \\
\hline
 & (11-\omega_c^2)s+6(k+1-\omega_c^2)
\end{array}$$

$$\Rightarrow A(s) = (s+6)(s^2+\omega_c^2) + \underbrace{(11-\omega_c^2)s+6(k+1-\omega_c^2)}_{\text{Residual}}$$

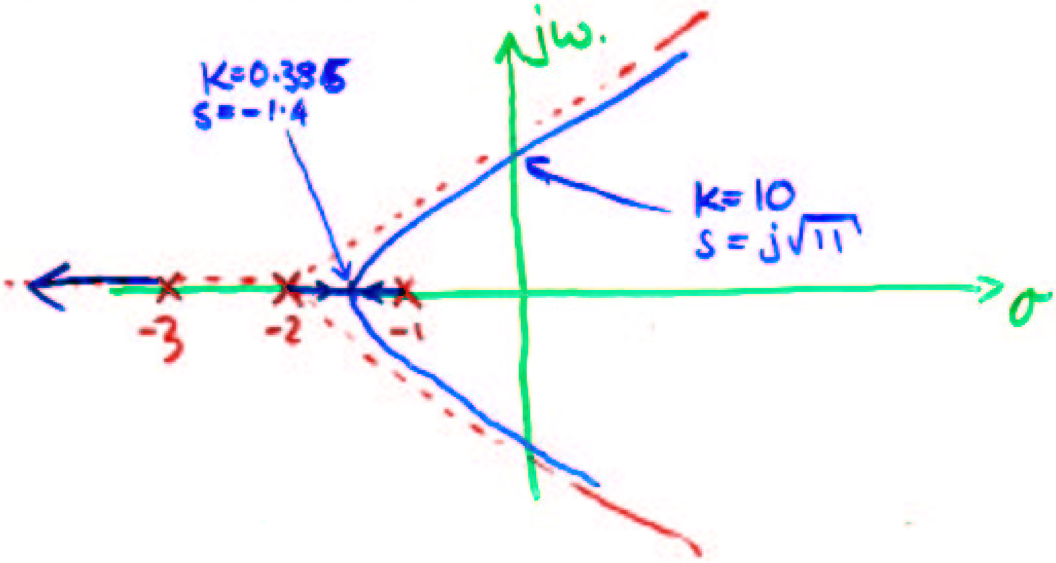
Poles on $j\omega$ axis if residual $\equiv 0$

ie. $11-\omega_c^2=0$

and $6(k+1-\omega_c^2)=0$

$$\Rightarrow \omega_c = \pm\sqrt{11}$$
$$\Rightarrow K = 10.$$

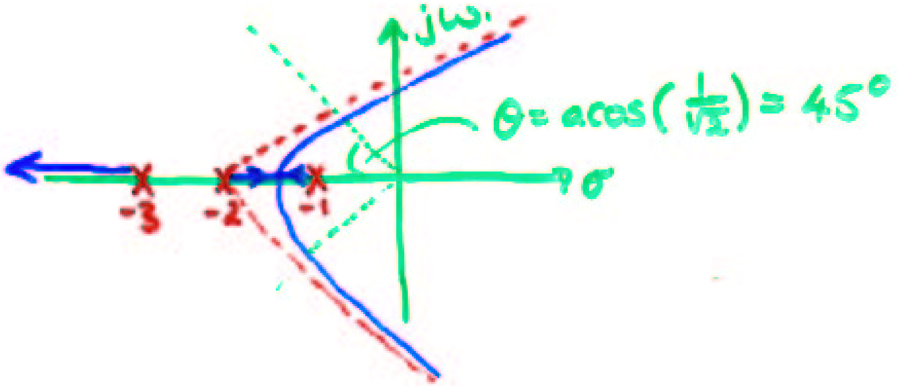
- Final sketch.



⇒ system is stable for all $0 \le K < 10$

More sophisticated design

Find K such that the damping factor of the complex pair of poles is $\zeta = \frac{1}{\sqrt{2}}$



We know that the ~~poles~~ complex conjugate pair of poles

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

comes from the factor

$$s^2 + 2\zeta\omega_n s + \omega_n^2.$$

(standard second-order system)

If $\zeta = \frac{1}{\sqrt{2}}$ this factor is

$$s^2 + \sqrt{2}\,\omega_n s + \omega_n^2.$$

The closed loop polynomial is

$$A(s) = s^3 + 6s^2 + 11s + 6(K+1)$$

Therefore, if closed loop is to have $\zeta = \frac{1}{\sqrt{2}}$, we must find K and ω_n such that

$$A(s) = (s^2 + \sqrt{2}\,\omega_n s + \omega_n^2)\,\tilde{A}(s)$$

Design approach is the same

① Polynomial division to find residual

$$\begin{array}{rl}
 & s + 6-\sqrt{2}w_n \\
s^2+\sqrt{2}w_n s+w_n^2 & \overline{\big)\, s^3 + 6s^2 + 11s + 6(k+1)} \\
 & -(s^3 + \sqrt{2}w_n s^2 + w_n^2 s + 0\ ) \\
\hline
 & 0 \quad (6-\sqrt{2}w_n)s^2 + (11-w_n^2)s + 6(k+1) \\
 & -\left[(6-\sqrt{2}w_n)s^2 + \sqrt{2}w_n(6-\sqrt{2}w_n)s + w_n^2(6-\sqrt{2}w_n) \right] \\
\hline
 & (w_n^2 - 6\sqrt{2}w_n + 11)s + \sqrt{2}w_n^3 - 6w_n^2 + 6(k
\end{array}$$

$$\Rightarrow A(s) = (s^2+\sqrt{2}w_n s+w_n^2)(s+6-\sqrt{2}w_n)$$
$$+ \underbrace{(w_n^2 - 6\sqrt{2}w_n + 11)s + \sqrt{2}w_n^3 - 6w_n^2 + 6(k+1)}_{\text{Residual}}$$

② Find w_n and K to set residual $= 0$

$$\Rightarrow w_n^2 - 6\sqrt{2}w_n + 11 = 0 \qquad Ⓐ$$

and

$$\sqrt{2}w_n^3 - 6w_n^2 + 6(k+1) = 0 \qquad Ⓑ$$

Solving Ⓐ yields $w_n \approx 6.884$ or 1.5969

Substituting into Ⓑ yields $K \approx -30$ or 0.5902.

Since root locus is for $0 \leq K < \infty$ the second solution is the one that makes sense.

Note that.

$$0 \leq K_{\substack{\text{poles}\\\text{coincide}}} \leq K_{\zeta=\frac{1}{\sqrt{2}}} < K_{\text{unstable}}.$$

$$0 < 0.385 < 0.59 < 10$$

Observe that the system can tolerate large gain + remain stable, but that if we also want to control the damping, the gain must be substantially smaller.