Design of Robust Pulse Shaping Filters via Semidefinite Programming

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Abstract — The design of a pulse shaping filter which provides maximal robustness to an unknown frequency-selective channel is formulated as a convex semidefinite programme, from which an optimal filter can be efficiently obtained. Robustness is measured by a sensitivity function derived from a bound on the probability of error. For unknown but bounded channels, the worst-case sensitivity is minimized, and for statistically modelled channels we minimize the average sensitivity. The formulation is used to design a 'chip' waveform with superior performance to that specified in a recent standard.

1 Introduction

In digital communications, waveform coding is often performed by linear pulse amplitude modulation (PAM) of a self-orthogonal ('root Nyquist') waveform, or an approximation thereof. The choice of waveform critically impacts many system performance criteria, and hence waveform design has been a topic of interest for many years. Although waveform design can be reduced to the design of a multi-rate finite impulse response (FIR) filter, typical design objectives (such as narrow 'bandwidth' and small intersymbol interference, ISI) are non-convex quadratic functions of the filter coefficients. That non-convexity can expose design algorithms to the intricacies of local minima. In previous work [1, 2], we reformulated the design of a spectrally-efficient orthogonal 'pulse shaping' filter as a convex semidefinite programme (SDP) [3] in the autocorrelation sequence of the filter. In that case, the bandwidth and intersymbol interference constraints become convex, and the globally optimal autocorrelation can be found using highly efficient interior point methods. An advantage of our formulation is that the transmission and reception filters can be computed directly from the output of the SDP, without auxiliary spectral factorization.

Although our previous work was focussed on distortionless transmission, we also showed [2, 4] that when sensitivity to timing error is measured in terms of the average mean square error of the data estimate, we obtain a convex quadratic function of the autocorrelation which can be efficiently incorporated into the SDP. That result can be extended to more general linear statistically modelled channels, but we will not do that here. Instead, we derive an alternative sensitivity measure from a bound on the probability of error. This measure involves linear and convex quadratic functions of the autocorrelation, and hence also can be efficiently incorporated into the SDP. We consider both statistically modelled channels and channels which are genuinely unknown, but are bounded. For the unknown but bounded channels, we minimize the worst-case sensitivity subject to a natural bound on performance in a distortionless channel, whereas for the statistically modelled channels we minimize the expected value of the sensitivity. As an example, we design a chip waveform with superior performance to that chosen in the IS95 standard [5] for Code Division Multiple Access mobile telephony.

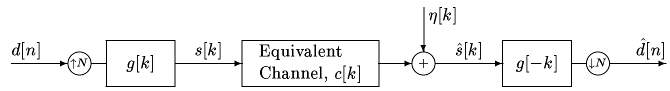

Fig. 1: Discrete-time model of baseband PAM.

2 Design framework

Consider the discrete-time baseband PAM scheme illustrated in Fig. 1, where the equivalent channel includes conversion to and from a continuous-time signal, carrier modulation and demodulation, and the physical frequency-selective (fading) channel. We focus on scenarios in which the equivalent channel does not vary significantly (in time) over the duration of an individual data symbol. In that case, the received data estimate $\hat{d}[n]$ is

$$\hat{d}[n] = f[0]d[n] + u[n] + \sum_k g[k - Nn]\eta[k], \qquad (1)$$

where $u[n] = \sum_{i\neq 0} f[i]d[n-i]$ is the ISI, $f[i] = \sum_k c[k]r_g[k - Ni]$, $r_g[m] = \sum_k g[k]g[k+m]$ is the autocorrelation function of $g[k]$, and $\eta[k]$ models the additive noise. (Here we have allowed finitely anti-causal filtering for convenience.)

A central design constraint in most applications is the spectral occupation (the 'bandwidth'), which is usually

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measured in terms of the (time-averaged) power spectrum. For stationary white data with zero mean and variance v_d, the power spectrum of $s[k]$ is $S_s(e^{j\omega}) = v_d\left|G(e^{j\omega})\right|^2$. In our designs we will constrain the spectral occupation by enforcing a relative spectral mask constraint expressed in decibels, (11). Note that $R_g(e^{j\omega}) = |G(e^{j\omega})|^2$, and hence that $S_s(e^{j\omega})$ is linear in $r_g[m]$ but is, in general, a non-convex quadratic function of $g[k]$.

The fundamental performance criterion of a PAM scheme is the probability of error, which we now evaluate in a simple case. We assume that $\eta[k]$ is zero mean, white and Gaussian, with variance N_0, and we normalize $g[k]$ so that it has unit energy ($r_g[0] = 1$). For binary equally-likely signalling with a transmitted signal energy per bit of E, and sign detection of $\hat{d}[n]$, the probability of error for a given value of ISI is

$$P_{e|u} = \frac{1}{4}\sum_{\xi=\pm1} \operatorname{erfc}\left(\left(f[0] + \xi u\right)\sqrt{E/N_0}\right), \qquad (2)$$

which is the average of $P_{e|u,d[n]=\pm1}$. The probability density function (pdf) of the ISI is

$$p_U(u) = \frac{1}{2^{(L_f-1)}} \sum_{v=1}^{2^{L_f-1}} \delta\left(u - \Delta^{(v)}\right),$$

where $L_f = \lfloor(2L_g + L_c - 3)/N\rfloor + 1$ is the length of $f[i]$,

$$\Delta^{(v)} = \sum_{i\neq0} f[i]d^{(v)}[n-i], \qquad (3)$$

and $d^{(v)}[n-i]$, $i \neq 0$, represents the vth combination of ±1's as the $L_f - 1$ interfering bits. Hence,

$$P_e = \frac{1}{2^{L_f+1}} \sum_{v=1}^{2^{(L_f-1)}} \sum_{\xi=\pm1} \operatorname{erfc}\left(\left(f[0] + \xi|\Delta^{(v)}|\right)\sqrt{E/N_0}\right). \quad (4)$$

By using Eq. (4) as an objective, it is possible to formulate a design problem for a filter which minimizes the probability of error for a given channel or class of channels. However, the mere evaluation of that objective, let alone its application in an optimization routine, has a computational cost which is exponential in L_f. In the next section, we attempt to bound P_e in such a way that we obtain efficient design algorithms for the filter which minimizes the bound.

3 Worst-case sensitivity

In this section, we seek a bound for the worst-case probability of error over a class of bounded channels. In passing, we will also show that the worst case P_e in a distortionless channel is bounded by the 'peak self-ISI' of the filter. We begin by observing that $\operatorname{erfc}(x)$ is a monotonically and rapidly decreasing function of x. Therefore that we have the following bound for the internal summation in Eq. (4), which holds for all v:

$$\begin{aligned}\sum_{\xi=\pm1} &\operatorname{erfc}\left(\left(f[0] + \xi|\Delta^{(v)}|\right)\sqrt{E/N_0}\right) \\ &\leq \operatorname{erfc}\left(f[0]\sqrt{E/N_0}\right) + \operatorname{erfc}\left(\left(f[0] - |\Delta^{(v)}|\right)\sqrt{E/N_0}\right).\end{aligned}$$

Furthermore, by applying the Hölder inequality,

$$|\Delta^{(v)}| \leq \max_j\left|d^{(v)}[j]\right|\sum_{i\neq0}\left|f[i]\right| = \sum_{i\neq0}\left|f[i]\right|, \qquad (5)$$

where the bound is achieved by at least one value of v. By expanding the expression for $f[i]$ in Section 2, we have

$$f[i] = r_g[0]c_e[Ni] + r_g[Ni] + \sum_{\ell\neq0} c_e[\ell + Ni]r_g[\ell],$$

where $c_e[k] = c[k] - \delta[k]$ is the 'error channel'. Applying the triangle and Cauchy-Schwartz inequalities,

$$\begin{aligned}\sum_{i\neq0}\left|f[i]\right| \leq \left|r_g[0]\right|\sum_{i\neq0}\left|c_e[Ni]\right| + \sum_{i\neq0}\left|r_g[Ni]\right| \\ + \sqrt{B_g}\sum_{i\neq0}\left(\sum_{\ell=-L_g+1}^{L_g-1}\left|c_e[\ell+Ni]\right|^2\right)^{1/2}, \end{aligned}\qquad (6)$$

where $B_g = \sum_{\ell\neq0} r_g[\ell]^2$. Using similar analysis,

$$f[0] \geq -\left|r_g[0]\right|\left|c_e[0]\right| - \sqrt{B_g}\left(\sum_{\ell=-L_g+1}^{L_g-1}\left|c_e[\ell]\right|^2\right)^{1/2}. \qquad (7)$$

Combining these results we have that

$$\begin{aligned}P_e \leq \frac{1}{4}&\operatorname{erfc}\left(f[0]\sqrt{E/N_0}\right) \\ &+ \frac{1}{4}\operatorname{erfc}\left(\left(f[0] - \sum_{i\neq0}|f[i]|\right)\sqrt{E/N_0}\right),\end{aligned} \qquad (8)$$

where $f[0]$ and $\sum_{i\neq0}|f[i]|$ are bounded by Eqs. (7) and (6), respectively.

We make two important observations from Eq. (8): (i) in a distortionless channel, P_e is bounded by an increasing function of the 'peak self-ISI' of $g[k]$, $\beta_g = \sum_{i\neq0}|r_g[Ni]|$; (ii) the worst-case P_e in a channel with a bounded two-norm is bounded by an increasing function of β_g and B_g. Hence, a natural objective is to design a filter which minimizes B_g subject to a bound on β_g, and a relative spectral mask constraint on $s[k]$. Unfortunately, B_g is a quartic polynomial in $g[k]$, and β_g and the power spectrum are, in general, non-convex quadratic functions of $g[k]$. This can expose the design algorithm to the intricacies of local minima.

In contrast, the spectral mask constraint and the constraint on β_g generate linear constraints on $r_g[m]$, and B_g is a convex quadratic function of $r_g[m]$. To complete the formulation of this design in terms of $r_g[m]$ instead of $g[k]$, we must add the additional constraint $R_g(e^{j\omega}) \geq 0$ for all $\omega \in [0, \pi]$, which is a necessary and sufficient condition for $r_g[m]$ to be factorizable in the form $r_g[m] = \sum_k g[k]g[k+m]$, (by the Féjer-Reisz Theorem). This is a semi-infinite constraint in that it must be satisfied for all $\omega \in [0, \pi]$. Although this constraint can be handled using discretization techniques, such an approach may lead to overly conservative designs and can be rather awkward numerically. As an alternative, we can apply the Positive Real Lemma [6] to transform this semi-infinite constraint into a finite dimensional linear matrix inequality. The Positive Real Lemma implies that $R_g(e^{j\omega}) \geq 0$ for all $\omega \in [0, \pi]$ if and only if there exists a real symmetric matrix $\boldsymbol{P}$ such that

$$\boldsymbol{M}(\boldsymbol{P}) \triangleq \begin{bmatrix} \boldsymbol{P} - \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} & \boldsymbol{c}^T - \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{b} \\ (\boldsymbol{c}^T - \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{b})^T & 2d - \boldsymbol{b}^T\boldsymbol{P}\boldsymbol{b} \end{bmatrix} \geq 0, \qquad (9)$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_{L-2} \\ 0 & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}, \qquad (10a)$$

$$\boldsymbol{c} = \begin{bmatrix} r_g[L-1] & \tilde{\boldsymbol{r}}_g \end{bmatrix}, \qquad d = 1/2, \qquad (10b)$$

and $\tilde{\boldsymbol{r}}_g = \big[r_g[L-2], r_g[L-3], \ldots, r_g[1]\big]$. By using this result, the design problem can be formulated as:

Formulation 1 *Given $\rho_\ell(\omega)$, $\rho_u(\omega)$, ϵ, N and L_g, find a filter of length L_g achieving* $\min \alpha$ *over $r_g[m]$, $m = 0, 1, \ldots, L_g - 1$, $\boldsymbol{P} = \boldsymbol{P}^T$, $\zeta > 0$ and $\alpha \geq 0$, subject to* $r_g[0] = 1$,

$$\zeta 10^{\rho_\ell(\omega)/10} \leq R_g(e^{j\omega}) \leq \zeta 10^{\rho_u(\omega)/10}, \text{ for all } \omega \in [0, \pi], \qquad (11)$$

$$\sum_{i=1}^{\lfloor (L_g-1)/N \rfloor} \big|r_g[Ni]\big| \leq \epsilon \qquad (12)$$

$$\sum_{\ell=1}^{L_g-1} r_g[\ell]^2 \leq \alpha \qquad (13)$$

and to Eq. (9) *holding for the realization in Eq.* (10)*, or show that none exist.*

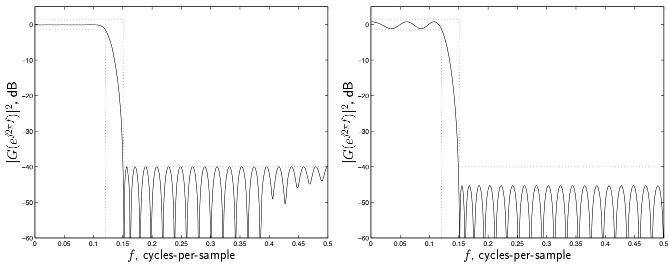
Although Eq. (11) is also a semi-infinite constraint, it is less 'critical' than $R_g(e^{j\omega}) \geq 0$ in the sense that a filter $g[k]$ with autocorrelation $r_g[m]$ may still exist, even if the mask is violated. Hence discretization of Eq. (11) over a sufficiently fine grid will usually suffice. Note also, that Eq. (12) can be re-written as a set of linear constraints with additional variables $\gamma_i \geq 0$, in the following standard manner: $-\gamma_i \leq r_g[Ni] \leq \gamma_i$ with $\sum_i \gamma_i \leq \epsilon$. Therefore, Formulation 1 consists of a linear objective, subject to linear inequality constraints (11) (12), a second-order cone [7] constraint (13), and a linear matrix inequality (9). Hence it is a convex 'symmetric cone programme' (of which SDPs are a special case), and the globally optimal autocorrelation sequence can be efficiently found via interior point methods. (`SeDuMi` [8] is a highly efficient MATLAB-based tool for this purpose.) Furthermore, a 'certificate of infeasibility' can be issued if the constraints cannot be satisfied by an autocorrelation sequence of the given length. An additional advantage of Formulation 1 is that the minimal phase optimal filter can be found directly from the optimal autocorrelation and the 'minimal' $\boldsymbol{P}$, without auxiliary spectral factorization [1, 2, 6]. (A simple modification to Formulation 1 ensures that the 'minimal' $\boldsymbol{P}$ is found [2].)

Pulse shaping filter design using Formulation 1 has an intuitively appealing interpretation in the frequency domain. Since $B_g = \int_{-\pi}^{\pi} \big(|G(e^{j\omega})|^2 - 1\big)^2 \, d\omega/(2\pi)$, minimizing B_g is equivalent to making $|G(e^{j\omega})|$ as flat as possible (in a mean-square sense). This interpretation also suggests a natural frequency-weighted design: minimize $\int_{-\pi}^{\pi} W(e^{j\omega})\big(|G(e^{j\omega})|^2 - 1\big)^2 \, d\omega$, for some real, non-negative, weighting function $W(e^{j\omega})$. To incorporate such a weighting into Formulation 1, we define $\boldsymbol{r}_g$ such that $[\boldsymbol{r}_g]_m = r_g[m]$, $m = 0, 1, \ldots, L_g - 1$, and then define $\boldsymbol{l}$ such that $\int_{-\pi}^{\pi} W(e^{j\omega})R(e^{j\omega}) \, d\omega = \boldsymbol{l}^T\boldsymbol{r}_g$ and $\boldsymbol{Q}$ such that $\int_{-\pi}^{\pi} W(e^{j\omega})R(e^{j\omega})^2 \, d\omega = \boldsymbol{r}_g^T\boldsymbol{Q}\boldsymbol{r}_g$. Finally, we replace the objective by $\alpha - 2\boldsymbol{l}^T\boldsymbol{r}_g$, and 'rotate' the second-order cone constraint (13) to $\|\boldsymbol{L}^T\boldsymbol{r}_g\|_2^2 \leq \alpha$, where $\boldsymbol{L}$ is an $L_g \times \text{rank}(\boldsymbol{Q})$ matrix such that $\boldsymbol{L}\boldsymbol{L}^T = \boldsymbol{Q}$. This frequency weighted version of Formulation 1 remains a symmetric cone programme and can also be efficiently solved.

Example 1 The filter specified for the synthesis of the chip waveform in IS95 [5] has $L = 48$ and $N = 4$. As an alternative to that filter, we designed a filter of the same length which minimizes B_g subject to $\beta_g \leq 0.02$ and the IS95 spectral mask, using Formulation 1. The power spectrum of the resulting filter is shown in Fig. 2, along with that of the IS95 filter. The improved frequency-flatness of the designed filter is clear from that figure. To demonstrate the performance improvement of the robust filter, we simulated the 'chip error rate' (CER) for transmission of binary chips over a slowly-varying linear channel with additive white Gaussian noise and sign detection of the chips at the receiver. The linear time-invariant 'snap shots' of the channel were of length 11 and were generated with $c[0] = 1$ and the remaining $c[k]$ being independent and Gaussian with zero mean and variance 0.1. (Such channels exhibit a wide variety of frequency selective effects.) The resulting CER curves, averaged over 1000 channel realizations, are plotted in Fig. 3, from which the improved performance of the robust filter is clear. For filters of this length, the spectral mask and self-ISI constraints are so 'tight' that the effects of a frequency-weighted sensitivity on the error rate are negligible. However, those effects become appreciable for longer filters. □

4 Average sensitivity

As an alternative to minimizing a worst-case sensitivity over a class of bounded channels, we might attempt to minimize an average sensitivity over a statistically modelled channel, where the channel taps $c[k]$ are random variables with known pdfs. Using the analysis in Sec-



(a) Robust filter. (b) IS95 filter.

Fig. 2: Relative power spectra (in decibels) of the filters in Example 1, with the spectral mask from IS95.

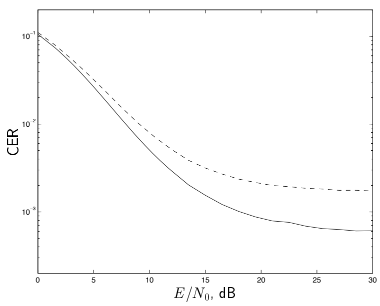


Fig. 3: Simulated chip error rates (CER) against signal-to-noise ratio for transmission over the class of channels in Example 1, using the filters in that example. Legend—Solid: robust filter; Dashed: IS95 filter.

tion 3, a natural approach would be to attempt to maximize $\mathbb{E}\{f[0]\}$ and to minimize $\mathbb{E}\{\sum_{i\neq 0}|f[i]|\}$, where $\mathbb{E}\{\cdot\}$ denotes expectation over the channel coefficients. Unfortunately this results in an optimization criterion which is not amenable to a simple efficient solution. However, by applying the Cauchy-Schwartz inequality to Eq. (3) we obtain $\left|\Delta^{(v)}\right| \leq \sqrt{L_f - 1}\left(\sum_{i\neq 0} f[i]^2\right)^{1/2}$. Now,

$$\mathbb{E}\Big\{\sum_{i\neq 0} f[i]^2\Big\} = \boldsymbol{r}_g^T \mathbb{E}\{\boldsymbol{Q}_c\}\boldsymbol{r}_g, \tag{14}$$

and $\boldsymbol{Q}_c$ is a positive semidefinite matrix determined by the channel coefficients. Similarly, $\mathbb{E}\{f[0]\} = \mathbb{E}\{\boldsymbol{l}_c^T\}\boldsymbol{r}_g$, where $\boldsymbol{l}_c$ is also determined by the channel coefficients. Since Eq. (14) is a convex quadratic in $r_g[m]$, and $\mathbb{E}\{f[0]\}$ is linear in $r_g[m]$ then these costs can be efficiently incorporated into the SDP. For example, one might wish to minimize Eq. (14) subject to a bound on $\mathbb{E}\{f[0]\}$, or to directly minimize an appropriate linear combination of Eq. (14) and $\mathbb{E}\{f[0]\}$. That problem starts to resemble the weighted version of Formulation 1, but with different weights. However, the performance of the scheme in a distortionless channel is now determined implicitly, rather than by a specific constraint on β_g.

5 Conclusion

In this paper we formulated the design of a robust spectrally-efficient pulse shaping filter as a semidefinite programme (SDP), from which an optimal filter can be efficiently obtained. Our sensitivity measure was based on a bound on the probability of error for binary signalling, which was derived by applying the Hölder inequality twice. As indicated in Section 4, other sensitivity measures can be obtained by using different instances of the Hölder inequality, but these measures do not necessarily lead to design problems which can be efficiently solved. Spectral efficiency was measured in terms of a relative spectral mask, but 'energy bandwidth' constraints can be easily accommodated [1, 2].

An alternative measure of robustness is the mean square error (MSE) in the data estimate. By generalizing earlier work on the robustness to timing error [2, 4] we can formulate the design of a filter which minimizes the average MSE for a statistically modelled channel as an SDP. Minimization of the worst-case MSE in a class of bounded channels can also be incorporated into the SDP by (significantly) modifying some analysis in [9].

In closing, we point out that we have sought robustness to unknown, but linear, channels. An interesting direction for further work is to examine ways in which robustness to channels with (unknown) non-linearities can be incorporated into the design framework.

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