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On non-central $\mathcal{H}_\infty$ controllers

D. Mustafa* and T.N. Davidson†
Department of Engineering Science
University of Oxford, Parks Rd, Oxford, OX1 3PJ, UK
email: Denis.Mustafa@eng.ox.ac.uk

Abstract

It is demonstrated that non-central $\mathcal{H}_\infty$ controllers can have advantages over the usual central $\mathcal{H}_\infty$ controller. In particular, it is shown that non-central controllers can improve the quadratic performance of the closed loop. It is further shown how to characterize non-central controllers with lower order than the central controller. The analysis is made possible by a state-space representation of the free contraction based on the Bounded Real Riccati equation.

1 Introduction

For a given generalized plant G and $\mathcal{H}_\infty$-norm bound γ, it was shown in [1] that the set of all proper, real-rational, stabilizing controllers K satisfying $\|\mathcal{F}_l(G,K)\|_\infty < \gamma$ is given by $K = \mathcal{F}_l(M_\infty, Q)$. That is, all feasible K's are given by the lower linear fractional map of a fixed transfer matrix M_∞ (calculated from the plant data) and the *free contraction* Q which is an arbitrary stable, proper, real-rational transfer matrix satisfying $\|Q\|_\infty < \gamma$. To date, little work has been done on how best to choose Q. Indeed the *central* $\mathcal{H}_\infty$ controller ($Q = 0$) is almost invariably chosen as a natural, simple choice that is also entropy minimizing, [2]. It is the purpose of the present work to show that in fact there may be genuine reasons for using *non-central* $\mathcal{H}_\infty$ controllers ($Q \neq 0$).

Previous work on the choice of Q has been hindered because the absence of a closed formula for $\|Q\|_\infty$ made the constraint $\|Q\|_\infty < \gamma$ somewhat awkward to deal with in state space. We avoid that difficulty by exploiting a recent state-space representation of the set of all permissible Q's which was developed in [3] for a different problem. The state-space matrices of Q are written in a useful unconstrained way which is the key to completing the analysis.

Firstly we consider choosing Q to reduce the LQG cost. We show that ordinarily non-central $\mathcal{H}_\infty$ controllers with only one extra state can be chosen to give smaller LQG cost than the central $\mathcal{H}_\infty$ controller. This is despite the fact that the central controller minimizes the closed-loop entropy which is known [2] to be a close upper bound on the LQG cost. Secondly, we investigate how Q can be chosen to force state cancellations in the natural realization of $K = \mathcal{F}_l(M_\infty, Q)$, leading to equations for reduced-order controllers.

After submission of the present paper, the independent paper [4] appeared, which also uses the parameterization of [3] to study mixed LQG/$\mathcal{H}_\infty$ control.

2 Preliminaries

The setup, notation and basic assumptions are as in [1]. The generalized plant G is taken to be of the form

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right],$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, $C_1 \in \mathbb{R}^{p_1\times n}$ and $C_2 \in \mathbb{R}^{p_2\times n}$, and is G assumed to satisfy the usual assumptions in [1]. Given a desired $\mathcal{H}_\infty$-norm bound γ, the two associated $\mathcal{H}_\infty$ Riccati equations are

$$X_\infty A + A^T X_\infty + X_\infty\left(\gamma^{-2}B_1B_1^T - B_2B_2^T\right)X_\infty + C_1^TC_1 = 0, \quad (1)$$

$$Y_\infty A^T + AY_\infty + Y_\infty\left(\gamma^{-2}C_1^TC_1 - C_2^TC_2\right)Y_\infty + B_1B_1^T = 0. \quad (2)$$

In [1] it is shown that there exists a stabilizing controller K such that $\|\mathcal{F}_l(G,K)\|_\infty < \gamma$ if and only if (1) and (2) have positive semi-definite stabilizing solutions satisfying $\rho(X_\infty Y_\infty) < \gamma^2$, where ρ is the spectral radius. If these conditions are satisfied, the set of all such controllers is given by $K = \mathcal{F}_l(M_\infty, Q)$, where $Q \in \mathcal{RH}_\infty$ and $\|Q\|_\infty < \gamma$, and

$$M_\infty = \left[\begin{array}{c|cc} \hat{A}_\infty & Z_\infty Y_\infty C_2^T & Z_\infty B_2 \\ \hline -B_2^TX_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right] =: \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & I \\ \hat{C}_2 & I & 0 \end{array}\right],$$

$$\hat{A}_\infty = A + \left(\gamma^{-2}B_1B_1^T - B_2B_2^T\right)X_\infty - Z_\infty Y_\infty C_2^TC_2,$$

$$Z_\infty = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}.$$

Ordinarily Q is set to zero but we wish to explore the effect of nonzero strictly proper Q. To represent the set of all possible strictly proper Q's by unconstrained parameters consider

$$\mathcal{Q}_\gamma^* := \{Q(s) : \|Q(s)\|_\infty < \gamma,\ Q(s) \text{ stable, real rational, strictly proper, of dimension } m_2\times p_2 \text{ and of McMillan degree} \le n_Q\}$$

and

$$\mathcal{Q}_\gamma := \Big\{C_Q(sI - A_Q)^{-1}B_Q : B_Q \in \mathbb{R}^{n_Q\times p_2},\ C_Q \in \mathbb{R}^{m_2\times n_Q},\ A_Q = A_s + A_{sk},\ A_{sk} = -A_{sk}^T \in \mathbb{R}^{n_Q\times n_Q},\ A_s = -B_QB_Q^T/(2\gamma^2) - C_Q^TC_Q/2\Big\}. \quad (3)$$

In [3] it is shown that $\mathcal{Q}_\gamma^* = \mathcal{Q}_\gamma$ by manipulating the Bounded Real Riccati equation. Hence in the sequel instead of the transfer function characterization $Q \in \mathcal{Q}_\gamma^*$ we use the explicit state-space characterization $Q \in \mathcal{Q}_\gamma$.

3 LQG performance

As $\mathcal{H}_\infty$ control theory has matured, mixed objective problems have received increasing attention. The particular problem of minimizing the LQG cost subject to an $\mathcal{H}_\infty$-norm bound has not yet been solved (some necessary conditions are given in [5]) but solutions to related problems have appeared in [6, 2, 7, 8, 9].

It is not our aim to solve any particular optimal problem here but rather we wish to point out that $Q = 0$ does not necessarily minimize the LQG cost subject to the $\mathcal{H}_\infty$-norm bound, even though $Q = 0$ minimizes the closely related entropy [2]. To substantiate that claim, consider the case where the controller is SISO (i.e., $m_2 = p_2 = 1$). Let Q have one state, that is $Q(s) = Q_1(s) = b_qc_q/(s - a_q)$, and let J be the usual LQG cost, $J := \|\mathcal{F}_l(G,K)\|_2^2$. Firstly we will show that with $b_q = \epsilon$ and $c_q = m\epsilon$, J can be written as

$$J = J_0 + \epsilon^2 J_2(m) + O(\epsilon^4),$$

where J_0 is the central LQG cost. We will then show that in general $J_2(m)$ can be made negative by suitable choice of m, giving $J < J_0$ for small enough ϵ. Thus the LQG cost can be improved using the one extra state in Q.

Letting $b_q = \epsilon$ and $c_q = m\epsilon$ and defining $\alpha = (m^2 + \gamma^{-2})/2$, the natural realization of the closed loop, $F(s) = \mathcal{F}_l(G, \mathcal{F}_l(M_\infty, Q_1))$, is

$$F(s) = \left[\begin{array}{cc|c} \tilde{A} & m\epsilon\tilde{B}_2 & \tilde{B}_1 \\ \epsilon\tilde{C}_2 & -\epsilon^2\alpha & \epsilon D_{21} \\ \hline \tilde{C}_1 & m\epsilon D_{12} & 0 \end{array}\right],$$

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where

$$\tilde{A} = \begin{bmatrix} A & B_2\hat{C}_1 \\ \hat{B}_1 C_2 & \hat{A} \end{bmatrix}, \ \tilde{B}_1 = \begin{bmatrix} B_1 \\ \hat{B}_1 D_{21} \end{bmatrix}, \ \tilde{B}_2 = \begin{bmatrix} B_2 \\ \hat{B}_2 \end{bmatrix},$$

$$\tilde{C}_1 = \begin{bmatrix} C_1 & D_{12}\hat{C}_1 \end{bmatrix}, \ \tilde{C}_2 = \begin{bmatrix} C_2 & \hat{C}_2 \end{bmatrix}.$$

With inspiration from [10], apply the block-diagonalizing transformation $\begin{bmatrix} I & \epsilon Y \\ Z & \epsilon \end{bmatrix}$ where

$$\tilde{A}Y + \epsilon^2\alpha Y + \epsilon^2 Y\tilde{C}_2 Y - m\tilde{B}_2 = 0, \qquad (4)$$
$$Z\tilde{A} + \epsilon^2\alpha Z - mZ\tilde{B}_2 Z + \epsilon^2\tilde{C}_2 = 0. \qquad (5)$$

This gives

$$F(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array}\right] \qquad (6)$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & \bar{A}_{22} \end{bmatrix} := \begin{bmatrix} \tilde{A} + \epsilon^2 Y\tilde{C}_2 & 0 \\ 0 & mZ\tilde{B}_2 - \epsilon^2\alpha \end{bmatrix} \\ \bar{B} &= \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} := \begin{bmatrix} \tilde{B}_1 + \epsilon^2 Y D_{21} \\ Z\tilde{B}_1 + \epsilon^2 D_{21} \end{bmatrix} \\ \bar{C} &= \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \\ \bar{C}_1 &:= \tilde{C}_1 + (\tilde{C}_1 Y - mD_{12})Z/(1 - ZY) \\ \bar{C}_2 &:= -(\tilde{C}_1 Y - mD_{12})/(1 - ZY). \end{aligned}$$

It is easy to verify by substitution into (4) and (5) that

$$\begin{aligned} Y &= mY_0 + \epsilon^2 Y_2 + O(\epsilon^4), \\ Z &= \epsilon^2 Z_2 + O(\epsilon^4), \end{aligned}$$

where Y_0 and Z_2 are independent of m, and hence that

$$(1 - ZY)^{-1} = 1 + \epsilon^2 mZ_2Y_0 + O(\epsilon^4).$$

Therefore,

$$\begin{aligned} \bar{A}_{11} &= \tilde{A} + \epsilon^2 mY_0\tilde{C}_2 + O(\epsilon^4), \\ \bar{A}_{22} &= \epsilon^2(mZ_2\tilde{B}_2 - \alpha) + O(\epsilon^4), \\ \bar{B}_1 &= \tilde{B}_1 + \epsilon^2 mY_o D_{21} + O(\epsilon^4), \\ \bar{B}_2 &= \epsilon^2(Z_2\tilde{B}_1 + D_{21}) + O(\epsilon^4), \\ \bar{C}_1 &= \tilde{C}_1 + \epsilon^2 m(\tilde{C}_1 Y_0 - D_{12})Z_2 + O(\epsilon^4), \\ \bar{C}_2 &= m(D_{12} - \tilde{C}_1 Y_0) + \epsilon^2(m^2(D_{12} - \tilde{C}_1 Y_0)Z_2Y_0 - \tilde{C}_1 Y_2) + O(\epsilon^4), \end{aligned}$$

Using these it can be verified that the controllability gramian, $\bar{P}$, of $F(s)$ in (6) is of the form

$$\bar{P} = \begin{bmatrix} \tilde{P} & 0 \\ 0 & 0 \end{bmatrix} + \epsilon^2 \begin{bmatrix} mP_2 & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} + O(\epsilon^4), \qquad (7)$$

where $\tilde{P}$ is the controllability gramian of $(\tilde{A}, \tilde{B}_1)$. In (7), $\tilde{P}$, P_2, and P_{12} are all independent of m, and

$$P_{22} = \frac{(Z_2\tilde{B}_1 + D_{21})(Z_2\tilde{B}_1 + D_{21})^T}{m^2 - 2mZ_2\tilde{B}_2 + \gamma^{-2}}.$$

The LQG cost of $F(s)$ can be calculated by $J = \text{trace}(\bar{C}\bar{P}\bar{C}^T)$, and hence, after some straightforward algebra,

$$J = J_0 + \epsilon^2 J_2(m) + O(\epsilon^4),$$

where

$$J_2(m) = mx + \left(\frac{m^2\beta^2}{m^2 - 2mZ_2\tilde{B}_2 + \gamma^{-2}}\right). \qquad (8)$$

The constants x and β in (8) are defined by

$$\begin{aligned} x &:= \text{trace}(\tilde{C}_1 P_2 \tilde{C}_1^T) + 2\text{trace}\left(\tilde{C}_1(\tilde{P}Z_2^T - P_{12})(\tilde{C}_1 Y_0 - D_{12})^T\right) \\ \beta^2 &:= \left[(D_{12} - \tilde{C}_1 Y_0)(Z_2\tilde{B}_1 + D_{21})\right]^T \left[(D_{12} - \tilde{C}_1 Y_0)(Z_2\tilde{B}_1 + D_{21})\right] \end{aligned}$$

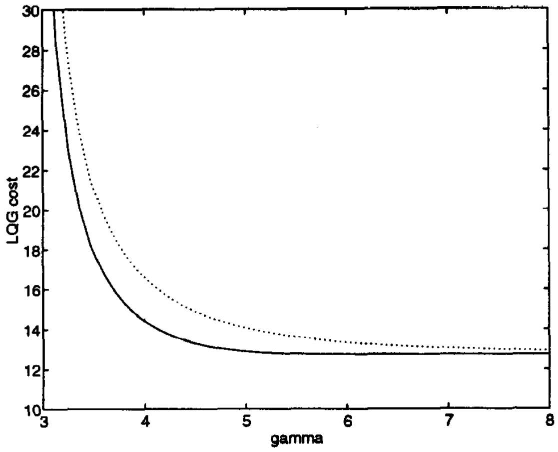

Figure 1: Variation of the LQG cost for the central (dotted) and one-state Q (solid) controllers with γ for the numerical example.

and are both independent of m. As $|m| \to \infty$ the term in brackets in (8) tends to β^2. Thus provided $x \neq 0$, $J_2(m)$ in (8) can be made negative by choosing m large enough with sign opposite to that of x. This makes the LQG cost smaller than the central cost for small enough ϵ.

The above analysis is illustrated with a simple example. Consider the plant

$$g(s) = \frac{-1.645s^3 - 1.03446s^2 - 0.04075s}{s^4 + 1.0603s^3 - 1.1154s^2 - 0.0565s - 0.0512},$$

taken from [11], and the Normalized $\mathcal{H}_\infty$ Control Problem for $g(s)$ (see [2, Chapter 7] for details), for which

$$G = \begin{bmatrix} \begin{bmatrix} g & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} g \\ 1 \end{bmatrix} \\ \begin{bmatrix} g & 1 \end{bmatrix} & \begin{bmatrix} g \end{bmatrix} \end{bmatrix} \ \text{so that} \ \mathcal{F}_l(G, K) = \begin{bmatrix} \frac{g}{1 - gK} & \frac{gK}{1 - gK} \\ \frac{gK}{1 - gK} & \frac{K}{1 - gK} \end{bmatrix}.$$

The above analysis implies that for all γ larger than the $\mathcal{H}_\infty$-optimal, the LQG cost can be reduced by adding an extra state in the controller. So for various values of γ the minimum LQG cost over admissable one-state Q was calculated via a combination of grid and gradient searches. (This optimization is not convex in the parameters b_q and c_q.) The results are plotted in Figure 1 and show that improvements of over 15% can be achieved for the price of an extra state in the controller.

4 Reduced-order controllers

It is easy to show that the natural realization for the controller $K = \mathcal{F}_l(M_\infty, Q)$ is

$$K = \left[\begin{array}{cc|c} \hat{A} & \hat{B}_2 C_Q & \hat{B}_1 \\ B_Q\hat{C}_2 & A_Q & B_Q \\ \hline \hat{C}_1 & C_Q & 0 \end{array}\right]. \qquad (9)$$

A reduced-order controller of order (at most) $n_K < n$ can be obtained if n_Q, A_Q, B_Q and C_Q can be found to make $n + n_Q - n_K$ states of K nonminimal, whilst satisfying $\|Q(s)\|_\infty < \gamma$. The procedure is outlined below. For a different approach to a similar problem see [12].

Firstly apply the state transformation $\begin{bmatrix} I_n & T \\ 0 & I_{n_Q} \end{bmatrix}$ to the realization of K in (9), where T is to be found. Then

$$K = \left[\begin{array}{cc|c} \hat{A} + TB_Q\hat{C}_2 & \hat{B}_2 C_Q + TA_Q - \hat{A}T - TB_Q\hat{C}_2 T & \hat{B}_1 + TB_Q \\ B_Q\hat{C}_2 & A_Q - B_Q\hat{C}_2 T & B_Q \\ \hline \hat{C}_1 & C_Q - \hat{C}_1 T & 0 \end{array}\right].$$

If A_Q, B_Q, C_Q and T are chosen such that

$$\hat{B}_1 + TB_Q = 0, \tag{10}$$

$$TA_Q - (\hat{A} - \hat{B}_1\hat{C}_2)T + \hat{B}_2 C_Q = 0, \tag{11}$$

then we have

$$K = \left[\begin{array}{cc|c} \hat{A} - \hat{B}_1\hat{C}_2 & 0 & 0 \\ B_Q\hat{C}_2 & A_Q - B_Q\hat{C}_2 T & B_Q \\ \hline \hat{C}_1 & C_Q - \hat{C}_1 T & 0 \end{array}\right]. \tag{12}$$

We now observe that

$$\hat{A} - \hat{B}_1\hat{C}_2 = A + \left(\gamma^{-2}B_1B_1^T - B_2B_2^T\right)X_\infty,$$

which is stable because, by assumption, (1) has a stabilizing solution. Hence the n uncontrollable modes associated with $\hat{A} - \hat{B}_1\hat{C}_2$ in (12) are all stable and can be cancelled leaving

$$K = \left[\begin{array}{c|c} A_Q - B_Q\hat{C}_2 T & B_Q \\ \hline C_Q - \hat{C}_1 T & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & 0 \end{array}\right].$$

Since $A_K = A_Q - B_K(\hat{C}_2 T)$ it follows that (A_K, B_K) is uncontrollable if and only if (A_Q, B_Q) is uncontrollable. So it is pointless to try to further reduce the order of K by forcing (A_K, B_K) to be uncontrollable as that would simply introduce uncontrollable states in Q. The next step is therefore to make (A_K, C_K) unobservable.

Of the n_Q states in A_K let $n_\xi = n_Q - n_K$ be the number to be made unobservable. Partition A_Q as

$$A_Q = \begin{bmatrix} A_{Q_{11}} & A_{Q_{12}} \\ A_{Q_{21}} & A_{Q_{22}} \end{bmatrix} \tag{13}$$

where $A_{Q_{11}} \in \mathbb{R}^{n_K \times n_K}$ and $A_{Q_{22}} \in \mathbb{R}^{n_\xi \times n_\xi}$, and partition B_Q, C_Q and T compatibly as

$$B_Q = \begin{bmatrix} B_{Q_1} \\ B_{Q_2} \end{bmatrix}, \quad C_Q = \begin{bmatrix} C_{Q_1} & C_{Q_2} \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}. \tag{14}$$

Similarly,

$$A_K = \begin{bmatrix} A_{K_{11}} & A_{K_{12}} \\ A_{K_{21}} & A_{K_{22}} \end{bmatrix} = \begin{bmatrix} A_{Q_{11}} & A_{Q_{12}} \\ A_{Q_{21}} & A_{Q_{22}} \end{bmatrix} - \begin{bmatrix} B_{Q_1} \\ B_{Q_2} \end{bmatrix} \hat{C}_2 \begin{bmatrix} T_1 & T_2 \end{bmatrix},$$

$$B_K = \begin{bmatrix} B_{Q_1} \\ B_{Q_2} \end{bmatrix}, \quad C_K = \begin{bmatrix} C_{K_1} & C_{K_2} \end{bmatrix} = \begin{bmatrix} C_{Q_1} - \hat{C}_1 T_1 & C_{Q_2} - \hat{C}_1 T_2 \end{bmatrix}.$$

If we set

$$A_{K_{12}} = A_{Q_{12}} - B_{Q_1}\hat{C}_2 T_2 = 0, \tag{15}$$

$$C_{K_2} = C_{Q_2} - \hat{C}_1 T_2 = 0, \tag{16}$$

then (if $A_{K_{22}}$ is stable) we can remove the n_ξ unobservable modes associated with $A_{K_{22}}$ leaving

$$K = \left[\begin{array}{c|c} A_{Q_{11}} - B_{Q_1}\hat{C}_2 T_1 & B_{Q_1} \\ \hline C_{Q_1} - \hat{C}_1 T_1 & 0 \end{array}\right], \tag{17}$$

which has the desired number of states n_K (although, of course, some may be nonminimal).

The analysis so far has used well-established state-cancellation techniques and at first glance further progress appears to be hampered by the condition $\|Q\|_\infty < \gamma$. Further analysis is made possible by replacing the transfer function constraint $\|Q\|_\infty < \gamma$ by the equivalent but constructive state-space constraint $Q \in \mathcal{Q}_\gamma$ from (3). Then B_Q and C_Q become free parameters and

$$\begin{aligned} A_Q &= \begin{bmatrix} A_{Q_{11}} & A_{Q_{12}} \\ A_{Q_{21}} & A_{Q_{22}} \end{bmatrix} \\ &= \begin{bmatrix} A_{sk_{11}} & A_{sk_{12}} \\ -A_{sk_{12}}^T & A_{sk_{22}} \end{bmatrix} \\ &\quad -\frac{1}{2\gamma^2}\begin{bmatrix} B_{Q_1}B_{Q_1}^T & B_{Q_1}B_{Q_2}^T \\ B_{Q_2}B_{Q_1}^T & B_{Q_2}B_{Q_2}^T \end{bmatrix} - \frac{1}{2}\begin{bmatrix} C_{Q_1}^T C_{Q_1} & C_{Q_1}^T C_{Q_2} \\ C_{Q_2}^T C_{Q_1} & C_{Q_2}^T C_{Q_2} \end{bmatrix}, \end{aligned}$$

where A_{sk} is an arbitrary $n_Q \times n_Q$ skew symmetric matrix.

We can now rewrite the conditions (10), (11), (15) and (16) to give the following characterization of the reduced-order controller in (17): The n_K state reduced-order controller in (17) is obtained by finding $B_{Q_1} \in \mathbb{R}^{n_K \times p_2}$, $B_{Q_2} \in \mathbb{R}^{n_\xi \times p_2}$, $C_{Q_1} \in \mathbb{R}^{m_2 \times n_K}$, $T_1 \in \mathbb{R}^{n \times n_K}$, $T_2 \in \mathbb{R}^{n \times n_\xi}$, $A_{sk_{11}} = -A_{sk_{11}}^T \in \mathbb{R}^{n_K \times n_K}$ and $A_{sk_{22}} = -A_{sk_{22}}^T \in \mathbb{R}^{n_\xi \times n_\xi}$ satisfying

$$0 = \hat{B}_1 + T_1 B_{Q_1} + T_2 B_{Q_2} \tag{18}$$

$$\begin{aligned} \left(\hat{A} - \hat{B}_1\hat{C}_2\right)T_1 &= \hat{B}_2 C_{Q_1} + T_1\left(A_{sk_{11}} - B_{Q_1}B_{Q_1}^T/(2\gamma^2) - C_{Q_1}^T C_{Q_1}/2\right) \\ &\quad -T_2\left(B_{Q_2}B_{Q_1}^T/\gamma^2 - T_2^T\left(\hat{C}_2^T B_{Q_1}^T + \hat{C}_1^T C_{Q_1}\right)\right) \end{aligned} \tag{19}$$

$$\begin{aligned} \left(\hat{A} - \hat{B}_2\hat{C}_1\right)T_2 &= -T_2\left(B_{Q_2}\hat{C}_2 + T_2^T\hat{C}_1^T\hat{C}_1/2\right)T_2 \\ &\quad +T_2\left(A_{sk_{22}} - B_{Q_2}B_{Q_2}^T/(2\gamma^2)\right) \end{aligned} \tag{20}$$

$$\left(A_{sk_{22}} - B_{Q_2}B_{Q_2}^T/(2\gamma^2) - C_{Q_2}^T C_{Q_2}/2 - B_{Q_2}\hat{C}_2 T_2\right) \text{ is stable} \tag{21}$$

and then setting

$$A_{sk_{12}} = B_{Q_1}B_{Q_2}^T/(2\gamma^2) + \left(C_{Q_1}^T\hat{C}_1/2 + B_{Q_1}\hat{C}_2\right)T_2 \tag{22}$$

$$C_{Q_2} = \hat{C}_1 T_2 \tag{23}$$

Equations (18), (22) and (23) are simply equations (10), (15) and (16) with the parameterization of $Q \in \mathcal{Q}_\gamma$ and the partitioning of (13). Equations (19) and (20) can be derived from a partitioning of (11) using substitutions from equations (18), (22) and (23) and equation (21) is the requirement that $A_{K_{22}}$ be stable. The set of equations in (18)–(23) is a set of design equations that characterizes reduced-order $\mathcal{H}_\infty$ controllers. The parameters A_{sk}, B_Q and C_Q enter in an unconstrained manner. Issues relating to the existence and computation of solutions to the equations are currently under investigation.

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