Nonlinear and Linear Broadcasting With QoS Requirements: Tractable Approaches for Bounded Channel Uncertainties

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Abstract—We consider the downlink of a cellular system in which the base station employs multiple transmit antennas, each receiver has a single antenna, and the users specify certain quality of service (QoS) requirements. We study the design of robust broadcasting schemes that minimize the transmission power necessary to guarantee that the QoS requirements are satisfied for all channels within bounded uncertainty regions around the transmitter's estimate of each user's channel. Each user's QoS requirement is formulated as a constraint on the mean square error (MSE) in its received signal, and we show that these MSE constraints imply constraints on the received signal-to-interference-plus-noise ratio. Using the MSE constraints, we present a unified approach to the design of linear and nonlinear transceivers with QoS requirements that must be satisfied in the presence of bounded channel uncertainty. The proposed designs overcome the limitations of existing approaches that provide conservative designs or are only applicable to the case of linear precoding. Furthermore, we provide computationally efficient design formulations for a rather general model of bounded channel uncertainty that subsumes many natural choices for the uncertainty region. We also consider the problem of the robust counterpart to precoding schemes that maximize the fidelity of the weakest user's signal subject to a power constraint. For this problem, we provide quasi-convex formulations, for both linear and nonlinear transceivers, that can be efficiently solved using a one-dimensional bisection search. Our numerical results demonstrate that in the presence of bounded uncertainty in the transmitter's knowledge of users' channels, the proposed designs provide guarantees for a larger range of QoS requirements than the existing approaches that are based on bounded channel uncertainty models and require less transmission power to provide these guarantees.

Index Terms—Broadcast channel, channel uncertainty, minimax design, linear transceivers, Tomlinson–Harashima transceivers, quality of service constraints, robust transceiver design.

I. INTRODUCTION

THE design of wireless broadcasting schemes that satisfy the quality of service (QoS) requirements of the intended receivers (users) is of growing interest in interactive communi-

cation applications and, in particular, in the downlink of cellular systems with differentiated services. The provision of multiple antennas at the transmitter (base station) of the downlink enables the design of schemes that (attempt to) satisfy the users' QoS requirements by spatially precoding the users' data in order to mitigate the multiuser interference at the (disjoint) receivers. The availability of accurate channel state information (CSI) at the transmitter is important in such schemes, as it enables the mitigation of the interference experienced by the receivers as a result of channel propagation. For scenarios in which perfect CSI is available at the transmitter, the problem of designing a precoder that minimizes the transmitted power required to satisfy a set of QoS requirements has been considered in [1]–[6] for the case of linear precoding and in [7]–[11] for the case of nonlinear precoding.

In practical downlink systems, the CSI that is available at the transmitter is subject to a variety of sources of imperfection, such as estimation errors, channel quantization errors, and short channel coherence time. For example, in systems in which the receivers feed back their quantized CSI to the transmitter (e.g., [12]–[14]), the uncertainty in the CSI that is available at the transmitter is typically dominated by quantization errors. Downlink precoder design methods that assume perfect CSI are particularly sensitive to these uncertainties [12], [13], and this suggests that the design of downlink precoding schemes ought to incorporate robustness to channel uncertainty. There are several approaches to incorporating robustness. One is to pose a statistical model for the error in the transmitter's estimate of the channels and to optimize a statistical measure of the satisfaction of the QoS requirements, such as "on average" or "with a certain probability of outage," e.g., [15], [16]. This stochastic uncertainty model is often appropriate in systems with uplink–downlink reciprocity in which the transmitter estimates the channels on the uplink. Another approach is to consider a bounded model for the error in the transmitter's estimate of the channels and to constrain the design of the precoder so that the users' QoS requirements are satisfied for all channels admitted by this model. This bounded uncertainty model is useful for systems in which it is difficult to provide the transmitter with an accurate statistical model for the channel uncertainty. In particular, it is useful for systems in which the users feed back quantized channel measurements to the transmitter, as knowledge of the quantization codebooks can be used to bound the quantization error.

For the downlink of cellular systems in which each receiver has a single antenna and the QoS requirements are formulated as constraints on the signal-to-interference-plus-noise ratio

(SINR) of each user, the design of a linear precoder that minimizes the transmitter power required to guarantee that each user's QoS requirement is satisfied for all admissible channels was considered in [17] and [18]. Related designs based on a bounded model for the errors in the transmitter's estimate of the (deterministic) autocorrelation matrices of the channel were considered in [3], [4], and [19]. While the methods proposed in [17] and [18] provide tractable design formulations and significant improvements in performance over previous designs, those methods have two limitations. First, they are not directly applicable to nonlinear precoding schemes, such as Tomlinson–Harashima precoding (THP). Secondly, although the robust linear QoS problem formulated in (13) of [18] guarantees that the users' SINR requirements are satisfied for all admissible channels, the tractability of the optimization problem was an open problem; see also [20].[1] In order to obtain tractable designs, a conservative design approach was taken in [17] and [18], and that approach requires the SINR constraints to be satisfied for a superset of channels that subsumes the original bounded set of admissible channels.

In this paper, we address both these limitations by providing semidefinite program (SDP) formulations for the design of both linear and nonlinear downlink precoding schemes that minimize the transmitted power required to ensure that each user's QoS requirement is satisfied for all admissible channels, without expanding the admissible set. We formulate each user's QoS requirement as a constraint on the mean square error (MSE) in each user's received signal, and we show these MSE constraints imply constraints on the received SINR of each user. Since the QoS is measured in terms of the MSE, our approach is immediately applicable to nonlinear Tomlinson–Harashima precoding, and the resulting designs include those for linear precoding as a special case. Furthermore, the proposed designs (for the linear case) are obtained with lower computational cost than those based on SINR formulations of the QoS requirements in [17] and [18]. As pointed out to us by a reviewer, for the case of linear precoding, the problem that we have addressed has been concurrently investigated in [23]; see also [22]. Some related work on the design of robust broadcast precoding schemes for bounded channel uncertainties with MSE performance objectives appears in [24].

We will present a unified treatment of a rather general bounded model for the channel uncertainty that can represent uncertainty regions resulting from a variety of sources of imperfection, including channel quantization errors. The model naturally includes channel uncertainty regions that are described using intersection of multiple uncertainty sets, e.g., the interval constraints on the entries of each user's channel that would arise from scalar quantization. Although we will

demonstrate that the case of multiple uncertainty regions is computationally intractable, we will provide conservative formulations that are efficiently solvable and represent a natural extension of the design formulation for the case of uncertainties that are described by single uncertainty region.

The proposed design approaches can be extended to obtain efficiently solvable quasi-convex formulations of some related design problems. In particular, we consider the robust counterpart of the problem of maximizing the fidelity of the weakest user's signal (minimizing the largest MSE among the users). For precoding schemes that assume perfect CSI at the transmitter, this problem was studied for the case of linear precoding schemes in [5] and [6]. For the bounded channel uncertainty model, tractable conservative approaches to the robust counterpart of this problem for linear precoders were provided in [18] (for the case of SINR constraints), but the problem has remained open for the case of nonlinear precoding. We provide quasi-convex formulations of this robust minimax problem (for MSE constraints) for both nonlinear and linear precoding schemes. These formulations can be efficiently solved using a one-dimensional bisection search or by formulating the problem as a generalized eigenvalue problem; e.g., [25].

We also consider the problem of determining the largest uncertainty region for which the QoS requirements can be satisfied for all admissible channels using finite transmission power. This problem is of considerable interest in the design of quantization codebooks for quantized channel feedback schemes. In that case, one might wish to choose the rate of the channel quantization scheme to be large enough (and the quantization cells small enough) for it to be possible to design a robust precoder with finite power. We provide quasi-convex formulations of this problem, too.

Our numerical results demonstrate the effectiveness of the proposed approach. In particular, the proposed designs provide guaranteed satisfaction of a larger set of QoS requirements than existing approaches that are based on bounded channel uncertainty models, even when the QoS requirements are specified in terms of the SINRs, and that they expend less transmission power in satisfying these requirements.

Our notation is as follows. We will use boldface capital letters to denote matrices, boldface lower case letters to denote vectors, and medium weight lower case letters to denote individual elements. $\mathbf{A}^T$ and $\mathbf{A}^H$ denote the transpose and the conjugate transpose of the matrix $\mathbf{A}$, respectively. The notation $\|\mathbf{x}\|$ denotes the Euclidean norm of vector $\mathbf{x}$, while $\|\mathbf{A}\|$ denotes the spectral norm (maximum singular value) of the matrix $\mathbf{A}$, and $\mathrm{E}\{\cdot\}$ denotes the expectation operator. The term $\mathrm{tr}\,(\mathbf{A})$ denotes the trace of matrix $\mathbf{A}$, and for symmetric matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \geq \mathbf{B}$ denotes the fact that $\mathbf{A} - \mathbf{B}$ is positive semidefinite. In addition, some of our design formulations will take the form of a second-order cone program or a semidefinite program [26].

II. SYSTEM MODEL

We consider the downlink of a multiuser cellular communication system with N_t antennas at the transmitter and K users, each with one receive antenna. We consider systems in which Tomlinson–Harashima precoding is used at the transmitter for multiuser interference presubtraction; e.g., [27] and [28]. As

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[1]While this paper was under review, we became aware of some unpublished work [21] in which an iterative approach was proposed for solving this optimization problem in polynomial time. That approach was applied to the robust linear QoS problem in [22]. However, the approach in [21] does not provide a "closed-form" design formulation and requires a "black box" oracle procedure that incurs considerable computational cost [21]. In contrast, the conservative approach taken in [17] and [18] yields a closed-form semidefinite program formulation that can be solved using efficient interior point algorithms. A work that develops an optimal algorithm for this problem that is polynomial time and possesses a closed-form semidefinite program formulation is currently under preparation by the authors.
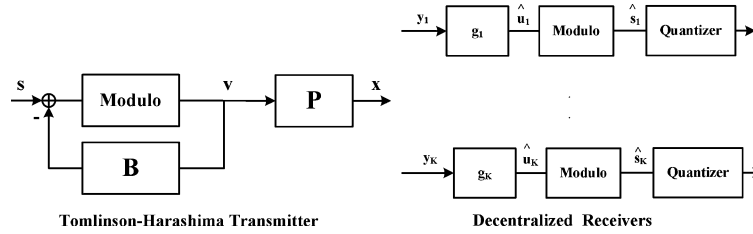
Fig. 1.   Multiple-input single-output downlink system with Tomlinson–Harashima precoding at the transmitter.
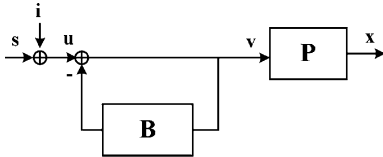

Fig. 2.   Equivalent linear model for the transmitter.

shown in Fig. 1, THP can be modelled using a feedback matrix $\mathbf{B} \in \mathbb{C}^{K \times K}$ and a feedforward precoding matrix $\mathbf{P} \in \mathbb{C}^{N_t \times K}$. Since linear precoding is the special case of the THP model in which $\mathbf{B} = \mathbf{0}$, we will focus our development on the THP case and will extract the special case results for linear precoding as they are needed.

The vector $\mathbf{s} \in \mathbb{C}^K$ in Fig. 1 contains the data symbol destined for each user, and we assume that s_k is chosen from a square quadrature amplitude modulation (QAM) constellation $\mathcal{S}$ with cardinality M and that $\mathrm{E}\{\mathbf{s}\mathbf{s}^H\} = \mathbf{I}$. The Voronoi region of the constellation $\mathcal{V}$ is a square whose side length is D.[2] In absence of the modulo operation, the output symbols of the feedback loop in Fig. 1, v_k, would be generated successively according to $v_k = s_k - \sum_{j=1}^{k-1} \mathbf{B}_{k,j} v_j$, where only the previously precoded symbols $v_1, \ldots, v_{k-1}$ are subtracted. Hence, $\mathbf{B}$ is a strictly lower triangular matrix. The role of the transmitter's modulo operation is to ensure that v_k remains within the boundaries of $\mathcal{V}$, and its effect is equivalent to the addition of the complex quantity $i_k = i_k^{re} D + j \, i_k^{\text{imag}} D$ to v_k, where $i_k^{re}, i_k^{\text{imag}} \in \mathbb{Z}$, and $j = \sqrt{-1}$. Using this observation, we obtain the standard linearized model of the transmitter that does not involve a modulo operation, as shown in Fig. 2; e.g., [28]. For that model

$$\mathbf{v} = (\mathbf{I} + \mathbf{B})^{-1} \mathbf{u} \qquad (1)$$

where $\mathbf{u} = \mathbf{i} + \mathbf{s}$ is the modified data symbol. As a result of the modulo operation, the elements of $\mathbf{v}$ are uncorrelated and uniformly distributed over the Voronoi region $\mathcal{V}$ [28, Th. 3.1] Therefore, the symbols of $\mathbf{v}$ will have slightly higher average energy than the input symbols $\mathbf{s}$. (This slight increase in the average energy is termed precoding loss [28].) For example, for square M-ary QAM, we have $\sigma_v^2 = \mathrm{E}\{|v_k|^2\} = \left(M/(M-1)\right)\mathrm{E}\{|s_k|^2\}$ for all k except the first one [28]. For moderate to large values of M, this power increase can be neglected and $\mathrm{E}\{\mathbf{v}\mathbf{v}^H\} = \mathbf{I}$ is often used; e.g.,

[27]–[29]. Hence, the average transmitted power constraint can be written as $\mathrm{E}_\mathbf{v}\{\mathbf{x}^H\mathbf{x}\} = \mathrm{tr}(\mathbf{P}^H\mathbf{P})$.

We will consider narrow-band signalling schemes; the signals received at each user y_k can be written as

$$y_k = \mathbf{h}_k \mathbf{x} + n_k = \mathbf{h}_k \mathbf{P}(\mathbf{I} + \mathbf{B})^{-1}\mathbf{u} + n_k \qquad (2)$$

where $\mathbf{h}_k \in \mathbb{C}^{1 \times N_t}$ is a row vector representing the channel gains from the transmitting antennas to the kth receiver and n_k represents the zero-mean additive white noise at the kth receiver, whose variance is $\sigma_{n_k}^2$. At each receiver, the equalizing gain g_k is used to obtain an estimate $\hat{u}_k = g_k \mathbf{h}_k \mathbf{P}(\mathbf{I}+\mathbf{B})^{-1}\mathbf{u} + g_k n_k$ of the modified data symbol u_k. Following this linear receive processing step, the modulo operation is used to obtain $\hat{\mathbf{s}}$. In terms of the modified data symbols, we can define the error signal

$$\hat{u}_k - u_k = (g_k \mathbf{h}_k \mathbf{P} - \mathbf{m}_k - \mathbf{b}_k)\mathbf{v} + g_k n_k \qquad (3)$$

where $\mathbf{m}_k$ and $\mathbf{b}_k$ are the kth rows of the matrices $\mathbf{I}$ and $\mathbf{B}$, respectively. When the integer i_k is eliminated by the modulo operation at the receiver, which occurs with high probability even at reasonably low SINRs (e.g., [28, pp. 129, 147–148]), the error signal in (3) is equivalent to $\hat{s}_k - s_k$. Using this error signal, the MSE of the kth user is given by

$$\begin{aligned} \mathrm{MSE}_k &= \mathrm{E}\{|\hat{u}_k - u_k|^2\} \\ &= \|g_k \mathbf{h}_k \mathbf{P} - \mathbf{m}_k - \mathbf{b}_k\|^2 + |g_k|^2 \sigma_{n_k}^2 \\ &= \left\| \left[g_k \mathbf{h}_k \mathbf{P} - \mathbf{m}_k - \mathbf{b}_k \quad g_k \sigma_k \right] \right\|^2 \end{aligned} \qquad (4)$$

and this will be the key metric in our designs.

III. Transceiver Design With MSE Constraints

In this paper, we will consider downlink scenarios in which each user has a quality of service constraint that is expressed in the form of an upper bound on its mean square error MSE_k. The formulation of QoS design problem in terms of the MSEs is motivated by the following result.

Lemma 1: Consider a scenario in which the users' channels are $\{\mathbf{h}_k\}_{k=1}^K$. If there exists a transceiver design $\mathbf{P}$, $\mathbf{B}$, g_k that guarantees that $\mathrm{MSE}_k \leq \zeta_k$, then that design guarantees that $\mathrm{SINR}_k \geq (1/\zeta_k) - 1$.

Proof: See Appendix A.   ∎

The statement in Lemma 1 implies that if we guarantee that the MSE is below a certain threshold for all channels in a given admissible set, then this implies a guarantee on the SINR for all channels in that admissible set, where the MSE and SINR are those obtained under the (common) assumption that the integer i_k is removed by the modulo operation at the receiver. This implication enables us to develop robust QoS designs based on

[2]The length of the side of the constellation D is equal to $\sqrt{M}d$, where d is the distance between two successive constellation points along either of the basis directions.

MSE constraints. As we will show in the remaining sections of the paper, doing so leads to designs with better performance, lower complexity, and broader applicability than the existing designs [4], [17]–[19], which are based on SINR constraints, even when the QoS constraints are specified in terms of SINR.

A. Perfect CSI Case

In order to facilitate our development of robust precoding schemes with QoS constraints, we will briefly consider the design problem for the case in which the transmitter has accurate knowledge of the users' channels. Iterative design approaches for the perfect CSI case have been considered in [8], [9], and [11] and the design problem was considered under zero-forcing criteria in [7] and [10]. Our approach to the perfect CSI case includes deriving a convex conic formulation of the Tomlinson–Harashima transceiver with QoS constraints. This formulation will enable us to develop robust counterparts for the case of bounded channel uncertainty and will allow the incorporation of different power constraints on the transmitter. In the case of perfect CSI, the design of the downlink transceiver components $\mathbf{P}$, $\mathbf{B}$, and g_k so as to minimize the total transmitted power subject to satisfying the users' MSE requirements can be formulated as

$$\min_{\mathbf{P}, \mathbf{B}, g_k} \quad \|\mathrm{vec}(\mathbf{P})\|^2 \tag{5a}$$

$$\text{subject to} \quad \left\| \begin{bmatrix} g_k \mathbf{h}_k \mathbf{P} - \mathbf{m}_k - \mathbf{b}_k, & g_k \sigma_k \end{bmatrix} \right\|^2 \leq \zeta_k. \tag{5b}$$

In the following lemma, we will show that g_k can be chosen to be real and positive without loss of generality.

Lemma 2: Consider the design problem in (5). If $\{|g_k|e^{j\theta_k}\}$, $\mathbf{P}$, and $\mathbf{B}$ are the optimal equalization gains, precoding matrix, and feedback matrix, respectively, then $\{|g_k|\}$, $\mathbf{P}\,\mathrm{Diag}(e^{j\theta_1}, \ldots, e^{j\theta_K}), \mathrm{Diag}(e^{-j\theta_1}, \ldots, e^{-j\theta_K})\,\mathbf{B}\,\mathrm{Diag}(e^{j\theta_1}, \ldots, e^{j\theta_K})$ are also optimal.

Proof: Consider the transceiver whose parameters are $\{|g_k|\}$, $\mathbf{P}\,\mathrm{Diag}(e^{j\theta_1}, \ldots, e^{j\theta_K}), \mathrm{Diag}(e^{-j\theta_1}, \ldots, e^{-j\theta_K})\,\mathbf{B}\,\mathrm{Diag}(e^{j\theta_1}, \ldots, e^{j\theta_K})$. Then, the left-hand side of the MSE constraint of the kth user in (5b) can be written as

$$
\left\| \begin{bmatrix} e^{j(\theta_1 - \theta_k)} & (|g_k|\mathbf{h}_k\mathbf{p}_1 e^{j\theta_k} - b_{k,1}) \\ \vdots & \\ e^{j(\theta_{k-1} - \theta_k)} & (|g_k|\mathbf{h}_k\mathbf{p}_{k-1} e^{j\theta_k} - b_{k,k-1}) \\ & (|g_k|\mathbf{h}_k\mathbf{p}_k e^{j\theta_k} - 1) \\ e^{j(\theta_{k+1} - \theta_k)} & (|g_k|\mathbf{h}_k\mathbf{p}_{k+1} e^{j\theta_k}) \\ \vdots & \\ e^{(j\theta_K - \theta_k)} & (|g_k|\mathbf{h}_k\mathbf{p}_K e^{j\theta_k}) \\ & |g_k|\sigma_k \end{bmatrix}^T \right\|
$$

$$
= \left\| \begin{bmatrix} (|g_k|e^{j\theta_k}\mathbf{h}_k\mathbf{p}_1 - b_{k,1}) \\ \vdots \\ (|g_k|e^{j\theta_k}\mathbf{h}_k\mathbf{p}_{k-1} - b_{k,k-1}) \\ (|g_k|e^{j\theta_k}\mathbf{h}_k\mathbf{p}_k - 1) \\ (|g_k|e^{j\theta_k}\mathbf{h}_k\mathbf{p}_{k+1}) \\ \vdots \\ (|g_k|e^{j\theta_k}\mathbf{h}_k\mathbf{p}_K) \\ |g_k|e^{j\theta_k}\sigma_k \end{bmatrix}^T \right\| \tag{6}
$$

where $\mathbf{p}_j$ is the jth column of $\mathbf{P}$. By extracting the unitary factor $\mathrm{Diag}(e^{j(\theta_1 - \theta_k)}, \ldots, e^{j(\theta_K - \theta_k)}, e^{-j\theta_k})$ and exploiting the unitary invariance of the 2-norm, we obtain the equality in (6). The right-hand side of (6) is the MSE of the kth user for the transceiver whose parameters are $\{|g_k|e^{j\theta_k}\}$, $\mathbf{P}$, and $\mathbf{B}$. Furthermore, both transceivers have the same total transmitted power. ∎

Using the result of Lemma 2 and the definitions

$$\underline{\mathbf{h}}_k = \begin{bmatrix} \mathrm{Re}\{\mathbf{h}_k\} & \mathrm{Im}\{\mathbf{h}_k\} \end{bmatrix} \tag{7}$$

$$\underline{\mathbf{P}} = \begin{bmatrix} \mathrm{Re}\{\mathbf{P}\} & \mathrm{Im}\{\mathbf{P}\} \\ -\mathrm{Im}\{\mathbf{P}\} & \mathrm{Re}\{\mathbf{P}\} \end{bmatrix} \tag{8}$$

$$\underline{\mathbf{b}}_k = \begin{bmatrix} \frac{\mathrm{Re}\{\mathbf{b_k}\}}{g_k} & \frac{\mathrm{Im}\{\mathbf{b_k}\}}{g_k} \end{bmatrix} \tag{9}$$

$$\underline{\mathbf{m}}_k = \begin{bmatrix} \mathrm{Re}\{\mathbf{m}_k\} & \mathrm{Im}\{\mathbf{m}_k\} \end{bmatrix} \tag{10}$$

$$f_k = \frac{1}{g_k} \tag{11}$$

where, by definition, $\mathrm{Im}\{\mathbf{m}_k\} = \mathbf{0}$, the design problem in (5) can be formulated as a convex Second Order Cone Program (SOCP)

$$\min_{\mathbf{P}, \underline{\mathbf{B}}, f_k, t} \quad t \tag{12a}$$

$$\text{subject to} \quad \|\mathrm{vec}(\underline{\mathbf{P}})\| \leq t \tag{12b}$$

$$\left\| \begin{bmatrix} \underline{\mathbf{h}}_k\underline{\mathbf{P}} - f_k\underline{\mathbf{m}}_k - \underline{\mathbf{b}}_k, & \sigma_{n_k} \end{bmatrix} \right\| \leq \sqrt{\zeta_k} f_k,$$
$$1 \leq k \leq K. \tag{12c}$$

This problem can be efficiently solved using general-purpose implementations of interior point methods [30], [31]; e.g., Se-DuMi [32]. An advantage of the convex conic formulation in (12) is the possibility to include shaping constraints (e.g., [18] and [33]) on the power transmitted from the antennas. These constraints can be expressed as either second-order cone or positive semidefiniteness constraints on the precoding matrix $\mathbf{P}$. The SOCP formulation can also incorporate multicell designs with per-cell power constraints on sets of antennas that belong to the same cell. These per-cell power constraints can also be formulated as second-order cone constraints on the elements of $\mathbf{P}$; see [24]. More importantly, however, the convex formulation in (12) enables us to derive robust counterparts of the original design problem in (5) for the uncertainty models presented in the following section.

IV. CHANNEL UNCERTAINTY MODEL

We will consider an additive uncertainty model of the form

$$\mathcal{U}_k(\delta_k, \mathbf{\Phi}_k, \mathbf{Q}_k) = \Bigg\{ \underline{\mathbf{h}}_k \mid \underline{\mathbf{h}}_k = \hat{\underline{\mathbf{h}}}_k + \underline{\mathbf{e}}_k = \hat{\underline{\mathbf{h}}}_k$$
$$+ \sum_{j=1}^{J} w_k^{(j)} \boldsymbol{\phi}_k^{(j)}, \ \mathbf{w}_k^T \mathbf{Q}_k \mathbf{w}_k \leq \delta_k^2 \Bigg\} \tag{13}$$

where $\hat{\mathbf{h}}_k$ is the transmitter's estimate of the kth user's channel and $\mathbf{e}_k$ is the corresponding error. The above model enables us to treat several different uncertainty regions in a unified way. For example, it can model the following uncertainty sets.

- *Ellipsoidal and Spherical Uncertainty Sets:* By choosing $\mathbf{Q}_k = \mathbf{I}$, the uncertainty set in (13) describes an ellipsoidal

uncertainty region around the channel estimate $\hat{\underline{\mathbf{h}}}_k$. The spherical uncertainty set with center $\hat{\underline{\mathbf{h}}}_k$ and radius δ_k is the special case that arises when $\boldsymbol{\Phi}_k$, the matrix whose rows are $\boldsymbol{\phi}_k^{(j)}$, is selected to be $\mathbf{I}_{2N_t}$.

- *Interval Uncertainty Sets:* Interval constraints on each element of $\underline{\mathbf{h}}_k$ can also be modeled as uncertainty sets of the form in (13). By taking $\boldsymbol{\Phi}_k$ to be $\mathbf{I}_{2N_t}$ and $\mathbf{Q}_k$ to be the matrix whose only nonzero element is $Q_{ii} = 1$, then the uncertainty set in (13) models an interval constraint on the ith entry of the error $\underline{\mathbf{h}}_k$. Interval constraints on multiple entries of $\underline{\mathbf{h}}_k$ can be represented as the intersection of uncertainty sets on the form (13); see Section V-A.

The additive uncertainty model in (13) is useful for systems in which the channel state information is (independently) quantized at each receiver and fed back to the transmitter; e.g., [12]. If a vector quantizer is employed at a given receiver, then the quantization cells in the interior of the quantization region can be often approximated by ellipsoids [34]. This ellipsoidal approximation can be substantially better than a spherical approximation when the channel coefficients are correlated, e.g., [35] and [36]. On the other hand, if a simple scalar quantizer is employed, the quantization regions can be modeled using a set of interval constraints.

V. TRANSCEIVER DESIGN WITH MSE CONSTRAINTS: UNCERTAIN CSI CASE

In this section, we will design a robust transceiver that minimizes the total transmitted power necessary to guarantee that the users' MSE requirements are satisfied for all admissible channels $\underline{\mathbf{h}}_k$ in the uncertainty region $\mathcal{U}_k(\delta_k)$ in (13). Using the formulation in (5), this robust problem can be stated as

$$\min_{\mathbf{P}, \underline{\mathbf{B}}, f_k, t} t \tag{14a}$$

$$\text{such that} \quad \|\operatorname{vec}(\underline{\mathbf{P}})\| \le t \tag{14b}$$

$$\|[\underline{\mathbf{h}}_k \underline{\mathbf{P}} - f_k \underline{\mathbf{m}}_k - \underline{\mathbf{b}}_k, \quad \sigma_{n_k}]\| \le \sqrt{\zeta_k} f_k$$
$$\forall \underline{\mathbf{h}}_k \in \mathcal{U}_k(\delta_k), 1 \le k \le K. \tag{14c}$$

This is a semi-infinite conic programming problem. In particular, the constraint (14c) represents K infinite sets of second-order cone (SOC) constraints, one for each $\underline{\mathbf{h}}_k \in \mathcal{U}_k(\delta_k)$. However, we can precisely characterize each of these infinite sets of SOC constraints using a single linear matrix inequality (LMI), as stated in the following formulation. (A derivation of this formulation is provided in Appendix B.) To state the formulation concisely, we define

$$\mathbf{a}_k = [\hat{\underline{\mathbf{h}}}_k \underline{\mathbf{P}} - \underline{\mathbf{m}}_k f_k - \underline{\mathbf{b}}_k, \quad \sigma_{n_k}] \tag{15}$$

$$\mathbf{A}_k(\zeta_k, \delta_k) = \begin{bmatrix} \sqrt{\zeta_k} f_k - \mu_k & \mathbf{0} & \mathbf{a}_k \\ \mathbf{0} & \mu_k \mathbf{Q}_k & \delta_k[\boldsymbol{\Phi}_k \underline{\mathbf{P}}, \ \mathbf{0}] \\ \mathbf{a}_k^T & \delta_k[\boldsymbol{\Phi}_k \underline{\mathbf{P}}, \ \mathbf{0}]^T & \sqrt{\zeta_k} f_k \mathbf{I} \end{bmatrix}. \tag{16}$$

1) Design Formulation 1: The robust transceiver design problem in (14) is equivalent to the following semidefinite program (SDP):

$$\min_{\substack{\boldsymbol{\mu}, t \\ \underline{\mathbf{P}}, \underline{\mathbf{B}}, f_k}} t \tag{17a}$$

$$\text{such that} \quad \|\operatorname{vec}(\underline{\mathbf{P}})\| \le t \tag{17b}$$

$$\mathbf{A}_k(\zeta_k, \delta_k) \ge \mathbf{0}, \quad 1 \le k \le K. \tag{17c}$$

$\square$

This result shows that the original design problem in (14) with an infinite set of constraints is equivalent to the convex SDP in (17), which can be efficiently solved using interior point methods, e.g., [32]. Such equivalence is an advantage of the structure of the uncertain parameter of the SOC representation, in (14c). In these SOC constraints, the channels $\underline{\mathbf{h}}_k$, and consequently the uncertain parameters, exist only on one side of the SOC. Hence, one can obtain an exact characterization of these uncertain SOCs that can be incorporated into the SDP. In contrast, when the QoS requirements are of the form of bounds on the SINR, then even in the case of linear precoding, both sides of the SOC constraints that enforce the QoS requirement depend on $\underline{\mathbf{h}}_k$, and the resulting design problem is substantially more difficult to solve [22]. In [17] and [18], a conservative approach to the robust design problem was taken to the QoS problem with SINR constraints, and this facilitated the reduction of the design problem to an SDP. As demonstrated by (17), for the case of MSE constraints, the robust QoS design problem can be efficiently solved without introducing conservatism.

A. Case of Intersecting Uncertainty Sets for Each $\mathbf{h}_k$

The formulation of the design problem in (14) extends naturally to the case in which the uncertainty region for each $\mathbf{h}_k$ is described as the intersection of more than one uncertainty set $\mathcal{U}_k^\ell$ of the form (13); that is, the uncertainty set is of the form

$$\tilde{\mathcal{U}}_k = \bigcap_{\ell=1}^L \mathcal{U}_k^\ell(\delta_k, \boldsymbol{\Phi}_k, \mathbf{Q}_k^\ell). \tag{18}$$

Note that there is no restriction in assuming that each $\mathcal{U}_k^\ell$ has the same uncertainty parameters δ_k and $\boldsymbol{\Phi}_k$, since $\mathbf{Q}_k^\ell$ in (13) can be chosen to accommodate different sizes and geometrical regions. Examples of constraint sets of the form in (18) include the interval constraints on each entry of $\underline{\mathbf{h}}_k$ that arise when scalar quantization is employed.

Although the design formulation involving uncertainty sets of the form (18) is a natural extension of that in (14), it can be shown, based on [37], that the resulting problem is NP-hard. In particular, the transformations that lead to the efficiently-solvable formulations of (14) [compare to (17)] do not extend to this case. However, by adopting a conservative approach, one can obtain an efficiently solvable approximation to the problem with the uncertainty set in (18). This conservative approach involves enveloping (18) in a superset that can be described more efficiently, and then requiring the MSE constraints to be satisfied for all channels in this superset. Using the derivation in Appendix B, one obtains the following conservative design formulation that has the same number of LMIs as that in (17). To state the formulation concisely, we define

$$\mathbf{B}_k(\zeta_k, \delta_k) = \begin{bmatrix} \sqrt{\zeta_k} f_k - \sum_{\ell=1}^L \mu_k^\ell & \mathbf{0} & \mathbf{a}_k \\ \mathbf{0} & \sum_{\ell=1}^L \mu_k^\ell \mathbf{Q}_k^\ell & \delta_k[\boldsymbol{\Phi}_k \underline{\mathbf{P}}, \ \mathbf{0}] \\ \mathbf{a}_k^T & \delta_k[\boldsymbol{\Phi}_k \underline{\mathbf{P}}, \ \mathbf{0}]^T & \sqrt{\zeta_k} f_k \mathbf{I} \end{bmatrix} \tag{19}$$

where $\mathbf{a}_k$ was defined in (15).

TABLE I
COMPARISON OF THE SIZES OF DESIGN FORMULATION 1 AND THAT OF THE STRUCTURED SDP APPROACH IN [18]

Method	Number of Variables	Number of Constraints		
		Linear	SOC (number; size)	LMI (number; size)
RLin-Correl Appr. [3], [4]	$K(N_t + 1)(2N_t + 3)$	$K(N_t + 1)(2N_t + 1)$	0	$K; 2N_t$
Structured SDP [18]	$K(K + 1)(2K + 3)$ $+2N_t K + 1$	$K(K + 1)(2K + 1)$	$1; 2N_t K + 1$	$K; 2(K + 1)(2N_t + 1)$
Design Form. 1 - Linear	$(2N_t + 2)K + 1$	0	$1; 2N_t K + 1$	$K; 2(K + 1) + J$
Design Form. 1 - THP	$(2N_t + K + 1)K + 1$	0	$1; 2N_t K + 1$	$K; 2(K + 1) + J$

1) Design Formulation 2: The solution of robust transceiver design problem in (14) for the intersection of uncertainty sets in (18) is upper bounded by the solution of the following SDP:

$$\min_{\substack{\boldsymbol{\mu}, t \\ \underline{\mathbf{P}}, \underline{\mathbf{B}}, f_k}} \quad t \tag{20a}$$

$$\text{such that} \quad \left\| \text{vec}(\underline{\mathbf{P}}) \right\| \leq t \tag{20b}$$

$$\mathbf{B}_k(\zeta_k, \delta_k) \preceq \mathbf{0}, \quad 1 \leq k \leq K. \tag{20c}$$

$\square$

B. Largest Feasible Uncertainty Size

In this section, we consider the related design problem of finding the largest value of the uncertainty size δ, namely, $\delta_{\max}$, for which there exists a robust transceiver of finite power that satisfies the MSE constraints for all admissible channels in the uncertainty region of size $\delta_{\max}$. This problem is connected to the problem of designing codebooks for the quantization of the users' channels. The codebook design must yield quantization regions that can be "covered" by uncertainty sets of size $\delta_{\max}$ in order for the robust transceiver design problem to be feasible. Using the problem formulation in (17), finding the value of $\delta_{\max}$ is equivalent to solving

$$\max_{\underline{\mathbf{P}}, \underline{\mathbf{B}}, f_k, \boldsymbol{\mu}, \rho} \quad \rho \tag{21a}$$

$$\text{such that} \quad \mathbf{A}_k(\zeta_k, \rho) \succeq \mathbf{0}, \quad 1 \leq k \leq K \tag{21b}$$

where $\mathbf{A}_k(\zeta_k, \rho)$ is defined in (16). Since ρ is an optimization variable rather than a design parameter, the bilinear terms in $\mathbf{A}_k(\zeta_k, \rho)$ mean that the design problem in (21) is not jointly convex in the design variables ρ and $\underline{\mathbf{P}}$. However, the problem is quasi-convex (see [26]), and an optimal solution can be efficiently found using a one-dimensional bisection search on ρ in which the problem solved at each step is the convex feasibility problem corresponding to (21) with a fixed value for ρ. For the case of the intersection of uncertainty regions in (18), the conservative constraint $\mathbf{B}_k(\zeta_k, \rho)$ in (19) may be used in place of (21b). In that case, the optimal value of the design problem becomes a lower bound on $\delta_{\max}$. It is worth observing that the largest uncertainty size for the special case of linear precoding is less than that of its THP counterpart. This follows by observing that finding $\delta_{\max}$ in the linear precoding case solves a restriction of (21) in which $\underline{\mathbf{B}}$ is set to $\mathbf{0}$.

C. Robust Broadcasting With QoS Requirements: MSE Versus SINR Constraints

In Section V, we presented design formulations for nonlinear and linear broadcasting transceivers with QoS requirements

under bounded channel uncertainty. The QoS requirements were formulated as MSE constraints. That design approach provides some attractive features when compared to the conservative design approaches in [17] and [18] in which the QoS requirements were formulated as constraints on the SINR. The work in [17] and [18] was restricted to the design of linear precoders and to uncertainty models consisting of a single spherical uncertainty region for each channel. Furthermore, in order to obtain a design algorithm with reasonable computational cost, a conservative design approach was taken. Beside being applicable to nonlinear THP schemes, the design approach of Section V provides exact efficiently solvable design formulations for a class of uncertainty models that encompasses many common uncertainty regions, including spherical regions. Furthermore, it enables a natural generalization to the case in which the uncertainty is described by multiple, and possibly different, intersecting regions. Lastly, the design approach proposed in Section V requires substantially less computational effort than that in [17] and [18].

In Table I, we provide comparisons of the sizes of the SDPs associated with Design Formulation 1, for both linear and nonlinear transceivers, and for that of the best conservative approach, namely the "structured SDP" approach in [18, Sect. IV]. (Table I also includes the corresponding data for the design approach in [3] and [4].) To assist in the comparison, we would like to point out that the dimension of the uncertainty ellipsoid J is less than or equal to the dimension of $\underline{\mathbf{e}}_k$, which is $2N_t$. For spherical uncertainty regions, $J = 2N_t$. For the case of linear precoding, Table I shows that when compared to the structured SDP approach in [17] and [18], the proposed approach involves fewer variables ($O(N_t K)$ instead of $O(K^3) + O(N_t K)$) and smaller LMIs ($O(N_t + K)$ instead of $O(N_t K)$). When compared to the approach in [3] and [4], the proposed approach involves fewer variables ($O(N_t K)$ instead of $O(N_t^2 K)$) and LMIs that will typically be of roughly the same order.

VI. ROBUST COUNTERPART OF FAIR MINIMAX TRANSCEIVER DESIGN

In the previous section, the focus was on the robust counterpart of the transceiver design problem that minimizes the total transmitted power subject to the satisfaction of the users' MSE constraints. In this section, we consider the related problem of minimizing the maximum MSE among all users subject to a transmitted power constraint, in scenarios with uncertain CSI. This design problem provides a notion of fairness amongst the users based on the value of their MSEs. While this problem has been considered in scenarios that assume perfect CSI in [5] and [6], we can formulate the robust counterpart of this design

problem under the channel uncertainty model in (13) as the following semi-infinite quasi-convex optimization problem:

$$\min_{\underline{\mathbf{P}}, \underline{\mathbf{B}}, f_k, \sqrt{\zeta_0}} \quad \sqrt{\zeta_0} \tag{22a}$$

$$\text{such that} \quad \left\| [\mathbf{h}_k \underline{\mathbf{P}} - f_k \underline{\mathbf{m}}_k - \underline{\mathbf{b}}_k, \quad \sigma_{n_k}] \right\| \leq \sqrt{\zeta_0} f_k$$

$$\forall \, \underline{\mathbf{h}}_k \in \mathcal{U}_k(\delta_k), \quad 1 \leq k \leq K \tag{22b}$$

$$\frac{1}{2} \text{tr}(\underline{\mathbf{P}}\underline{\mathbf{P}}^T) \leq P_{\text{total}}. \tag{22c}$$

Using the characterization in (17c) of the infinite set of SOC constraints in (22b), this design problem can be formulated as the following quasi-convex optimization problem:

$$\min_{\underline{\mathbf{P}}, \underline{\mathbf{B}}, f_k, \sqrt{\zeta_0}} \quad \sqrt{\zeta_0} \tag{23a}$$

$$\text{such that} \quad \mathbf{A}_k(\zeta_0, \delta_k) \geq 0, \quad 1 \leq k \leq K \tag{23b}$$

$$\left\| \text{vec}(\underline{\mathbf{P}}) \right\| \leq \sqrt{2 P_{\text{total}}}. \tag{23c}$$

This problem can be efficiently solved by using a bisection search on $\sqrt{\zeta_0}$ in which the problem solved at each step is the convex feasibility problem generated by (23) with a fixed value of $\sqrt{\zeta_0}$. Alternatively, we can observe that each constraint in (23b) can be written as

$$\sqrt{\zeta_0} \begin{bmatrix} f_k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & f_k \mathbf{I} \end{bmatrix}$$

$$- \begin{bmatrix} \mu_k & \mathbf{0} & -\mathbf{a}_k \\ \mathbf{0} & -\mu_k \mathbf{Q}_k & -\delta_k[\boldsymbol{\Phi}_k \underline{\mathbf{P}}, \ \mathbf{0}] \\ -\mathbf{a}_k^T & -\delta_k[\boldsymbol{\Phi}_k \underline{\mathbf{P}}, \ \mathbf{0}]^T & \mathbf{0} \end{bmatrix} \geq 0 \tag{24}$$

where $\mathbf{a}_k$ was defined in (15). Hence, (22) is equivalent to minimizing the largest generalized eigenvalue of a pair of (block diagonal) symmetric matrices that depend affinely on the decision variables [25], [38]—a problem that takes the general form

$$\min_{\mathbf{x}, \alpha} \quad \alpha \tag{25a}$$

$$\text{such that} \quad \alpha \mathbf{A}_1(\mathbf{x}) - \mathbf{A}_2(\mathbf{x}) \geq 0 \tag{25b}$$

$$\mathbf{A}_1(\mathbf{x}) \geq 0 \tag{25c}$$

$$\mathbf{B}(\mathbf{x}) \geq 0 \tag{25d}$$

where $\mathbf{B}(\mathbf{x})$ is an arbitrary LMI constraint on the decision variable $\mathbf{x}$. In our application, $\mathbf{B}(\mathbf{x}) \geq 0$ can be used to describe the power constraint and spatial shaping constraints; see [18] and [33]. This observation allows us to employ more efficient algorithms (e.g., [25] and [39]) that exploit the structure of the constituent matrices in (24).

VII. NUMERICAL STUDIES

In this section, we demonstrate the performance of the proposed robust QoS designs for nonlinear THP (RTHP-order 1, 2) and linear precoding (RLin) that were presented in Section V and provide comparisons with existing approaches that assume bounded channel uncertainty models. For THP, ordering of the users' channels is necessary prior to precoding. Finding the optimal ordering requires an exhaustive search over all possible permutations of the transmitter's estimate of the users' channels

$\hat{\underline{\mathbf{h}}}_k$, and instead of that we have implemented two suboptimal ordering methods. The first method applies the BLAST ordering in [40] to the transmitter's estimate of the users' channels. The second method is a generalization of the ordering method in [41] that selects a channel ordering that minimizes the reciprocals of the received SINRs when the precoder matrix $\mathbf{P}$ is an identity matrix. In our generalization, the ordering selection criterion is minimizing the sum of each user's SINR requirements divided by its received SINR (when $\mathbf{P} = \mathbf{I}$), a quantity that is proportional to the power necessary for each user to achieve its SINR requirement.

In our numerical studies, we consider a spherical uncertainty region of radius δ_k for each user. This model will facilitate the comparisons with other existing approaches for the linear precoding model, namely, the robust autocorrelation matrix approach in [3] and [4] (RLin-Correl Appr.), the robust power loading approach (RLin-PL1) using SINR constraints in [19], and the robust power loading approach (RLin-PL2) using MSE constraints in [42]. We will also provide comparisons with the conservative approach to robust linear precoding with SINR constraints in [17] and [18]. The work in [17] and [18] presented three conservative approaches, and we are comparing with the best conservative approach, namely, the structured SDP approach in [17, Sect. VI]; see also [18, Sect. IV]. As we make the comparisons, we would like to point out that the existing approaches to robust linear QoS precoding do not extend to Tomlinson–Harashima precoding, but the approaches proposed herein are inherently applicable to both linear and Tomlinson–Harashima precoders.

In order to totally specify the schemes used in our comparisons, we point out that the approaches in [19] and [42] require the beamforming vectors (normalized columns of $\mathbf{P}$) to be specified. We will use the zero-forcing beamforming vectors (the columns of the pseudoinverse of $\hat{\mathbf{H}}$). In addition, the approaches in [3], [4], and [19] are based on uncertainty models that are different from that in (13) and from each other. The approach in [3] and [4] considers a model in which the spectral norm of the error in the (deterministic) autocorrelation matrix $\mathbf{C}_k = \mathbf{h}_k^H \mathbf{h}_k$ is bounded, and in the approach in [19] the Frobenius norm of the error in $\mathbf{C}_k$ is bounded. However, by bounding these norms of $\mathbf{C}_k$ in terms of the norm of $\mathbf{e}_k$, we can obtain the smallest uncertainty set for $\mathbf{C}_k$ that contains all the channels in the set specified by $\|\mathbf{e}_k\| \leq \delta_k$. Furthermore, the admissible uncertainty $\mathbf{e}_k = \delta_k \hat{\mathbf{h}}_k / \|\hat{\mathbf{h}}_k\|$ lies on the boundaries of the uncertainty sets for $\mathbf{C}_k$ in [3], [4], and [19].[3] We will compare these schemes in an environment with $N_t = 3$ transmit antennas and $K = 3$ users. In our experiments, we will evaluate performance statistics for the standard case of independent Rayleigh fading channels in which the coefficients of the fading channels are mod-

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[3]A bound on the spectral norm of the error in the matrix $\mathbf{C}_k$ can be obtained as follows: $\|(\hat{\mathbf{h}}_k + \mathbf{e}_k)^H(\hat{\mathbf{h}}_k + \mathbf{e}_k) - \mathbf{h}_k^H \mathbf{h}_k\| = \|\hat{\mathbf{h}}_k^H \mathbf{e}_k + \mathbf{e}_k^H \hat{\mathbf{h}}_k + \mathbf{e}_k^H \mathbf{e}_k\| \leq \|\hat{\mathbf{h}}_k^H \mathbf{e}_k\| + \|\mathbf{e}_k^H \hat{\mathbf{h}}_k\| + \|\mathbf{e}_k^H \mathbf{e}_k\| = 2\|\hat{\mathbf{h}}_k\|\|\mathbf{e}_k\| + \|\mathbf{e}_k\|^2$. The same bound also holds for the Frobenius norm, since the matrices on the immediate right-hand side of the inequality are all rank one. Furthermore, the uncertainty $\mathbf{e}_k = \delta_k \hat{\mathbf{h}}_k / \|\hat{\mathbf{h}}_k\|$ achieves this upper bound with equality for both norms. Therefore, the chosen bound on $\mathbf{C}_k$ is the smallest (achievable) bound such that all the channels in the set specified by $\|\mathbf{e}_k\| \leq \delta_k$ lie in the uncertainty sets of the methods in [3], [4], and [19], and the admissible uncertainty $\mathbf{e}_k = \delta_k \hat{\mathbf{h}}_k / \|\hat{\mathbf{h}}_k\|$ lies on the boundaries of these sets. (See also [43].)
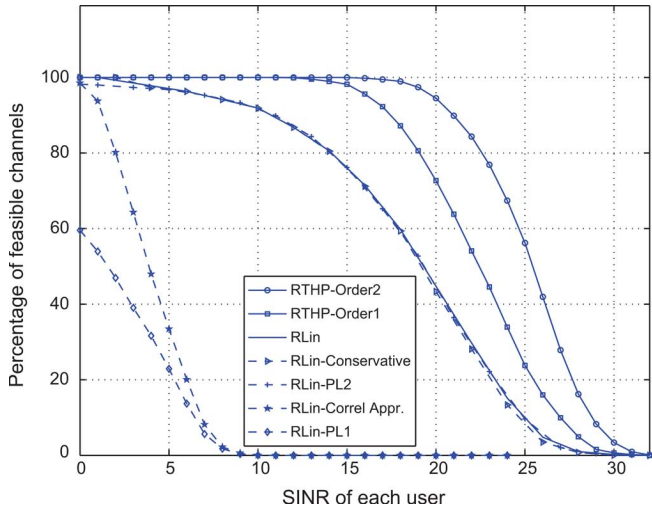
Fig. 3. Percentage of the 2000 channel realizations for which the robust QoS guarantee can be made against the required SINRs, for a system with $N_t = 3$ and $K = 3$.

eled as being independent circular complex Gaussian random variables with zero-mean and unit variance and the receivers' noise sources are modeled by zero-mean, additive, white, and circular Gaussians with unit variance.

A. Performance Comparisons Against SINR Requirements

In this comparison, we randomly generated 2000 realizations of the set of channel estimates $\{\hat{\mathbf{h}}_k\}_{k=1}^K$ and examined the performance of each method in the presence of uncertainties of equal sizes $\delta_k = \delta = 0.05, \forall k$. The SINR requirements of the three users are also equal. For each set of channel estimates and for each value of the required SINR, we determined whether each design method is able to generate a precoder (of finite power) that guarantees the required SINRs.

In Fig. 3, we plot the fraction of the 2000 channel realizations for which each method generated a precoder with finite power against the (equal) SINR requirements of the users. From this figure, it is clear that the proposed robust designs for linear (RLin) and nonlinear (RTHP-order 1, 2) precoding satisfy the SINR requirements for larger percentages of channels. The robust conservative approach for linear precoding (RLin-Conservative) [17], [18] and the power loading method in [42] achieve the QoS requirements for a percentage of channels that is quite close to that of the proposed linear approach (RLin). However, the proposed approach (RLin) has a significantly lower computational cost than the conservative approach (RLin-Conservative); see Table I. Furthermore, this approach is also applicable to nonlinear THP (RTHP-order 1, 2) and is applicable to richer set of uncertainties. For the robust linear power loading approach (RLin-PL2) in [42], the QoS design problem in terms of MSE constraints was justified as a heuristic measure for the SINR requirements. However, using Lemma 1, we showed that the MSE constraint of each user implies a minimum achieved SINR. Furthermore, there does not appear to be a direct extension of the power loading approach in [42] (nor that in [19]) to THP.
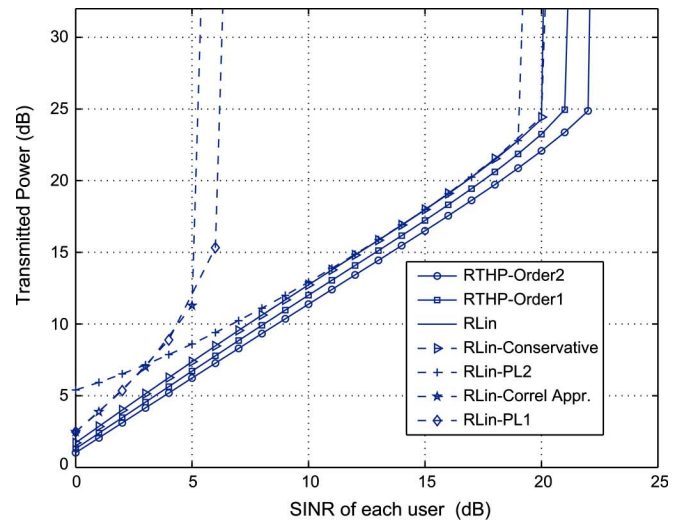
(a) Average transmitted power versus (equal) SINR requirements.
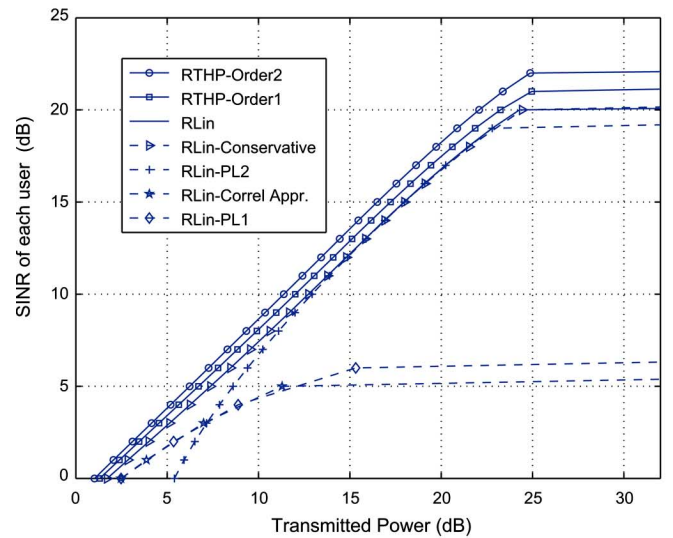
(b) Fair (equal) SINRs against the average transmitted power

Fig. 4. Relation between transmitted power and SINR requirements for the robust QoS design problem and the robust fair problem for a system with $N_t = 3$ and $K = 3$.

For the comparison in Fig. 4(a), we selected all the sets of channel estimates from the 2000 sets used in the previous experiment for which all design methods were able to provide robust QoS guarantees for all SINRs less than or equal to 6 dB. We calculated the average, over the 274 such channel environments, of the transmitted power required to achieve these robust QoS guarantees. We have plotted the average transmitted power versus the equal SINR requirement of each user in Fig. 4(a). This figure demonstrates the saturation effect that channel uncertainty imposes on the growth of the SINR of each user with the transmitted power for both of linear and nonlinear precoding. The SINR saturates at the value of SINR for which the method under consideration cannot provide the robust QoS guarantee with finite power for one (or more) of the channel estimates. A related effect was observed in [12] for nonrobust linear precoding on the MISO downlink with quantized CSI. Fig. 4(a) also illustrates the role that robust precoding can play
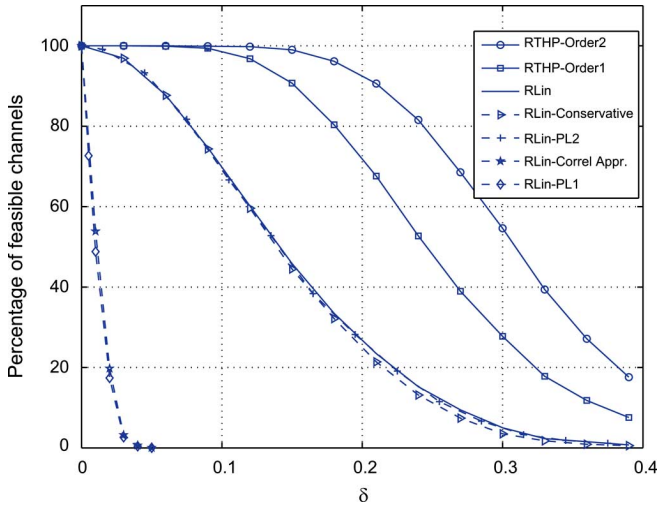
Fig. 5. Percentage of channel realizations for which the robust QoS guarantee can be made against the uncertainty size δ, for a system with $N_t = 3$ and $K = 3$.

Fig. 6. Average of the transmitted power $\text{tr}(\mathbf{P}^H \mathbf{P})$, on a linear scale, versus uncertainty size δ for a system with $N_t = 3$ and $K = 3$.

in extending the SINR interval over which linear growth with the transmitted power can be achieved. This is particularly evident for the robust nonlinear approaches (RTHP-order 1, 2) and the robust linear approach (RLin). We also observe that the second ordering method for Tomlinson–Harashima precoding provides better performance than the first one, since it selects the channel ordering in a way that attempts to minimize the sum of powers necessary to achieve each SINR requirement.

Since the previous experiments consider scenarios with equal SINR requirements for all users, by transposing the axes, these performance comparison curves can be interpreted as comparisons of different approaches for the robust fair broadcasting problem. This is to be expected because the problem of QoS design with equal requirements and the problem of max-min fair design are inverses of each other [6]. (The proof in [6] is directly extensible to the robust case.) As an example, in Fig. 4(b), we have computed the solution to the robust fair design in Section VI for the communications scenario of the second experiment, and it can be seen that it is the transposed version of Fig. 4(a).

B. Performance Comparisons Against Uncertainty Size

In this comparison, we used the 2000 randomly generated realizations of the set of channel estimates $\{\hat{\mathbf{h}}_k\}_{k=1}^K$ to examine the performance of each method in the presence of equal uncertainty $\delta_k = \delta, \forall k$. The QoS requirement of each user is such that the SINR is at least 10 dB. In Fig. 5, we provide the percentage of the 2000 channel realizations for which each method generated a precoder with finite power as a function of the size of the uncertainty. From this figure, it is clear that for a large range of uncertainty sizes, the proposed nonlinear approaches (RTHP-Order 1, 2) satisfy the SINR requirements for many more channel realizations than other approaches. This is due to the fact that the proposed linear approach is a special case of the proposed THP design, and the other existing linear approaches are either conservative or restricted to the optimization of powers for given transmit directions. While the performance
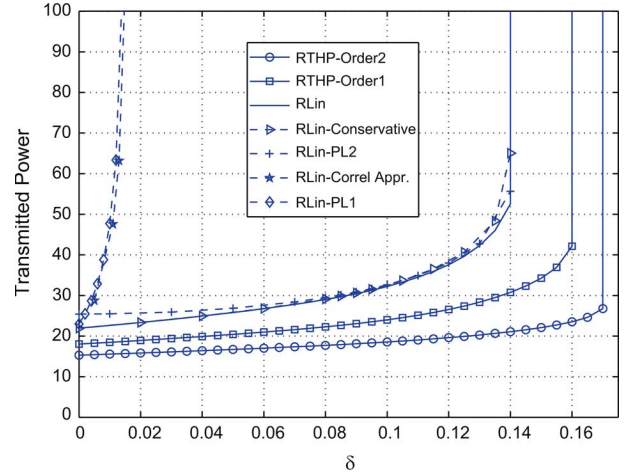
of the conservative linear precoding approach (RLin-conservative) in [17] and [18] and the robust linear power loading approaches (RLin-PL2) in [42] is quite close to that of the proposed linear design (RLin) in terms of number of channel realizations for which the method satisfies the robust (SINR-based) QoS requirements, they use more power in order to achieve the QoS requirements, as shown in Fig. 6.

In Fig. 6, we selected those sets of channel estimates from the 2000 sets used in the previous experiment for which all design methods were able to provide robust QoS guarantees for all uncertainties with $\delta \leq 0.015$. We calculated the average, over the 614 such channel environments, of the transmitted power required to achieve these robust QoS guarantees, and we have plotted the results for different values of δ in Fig. 6. The average transmitted power approaches infinity for a certain value of δ when for one (or more) of the channel estimates, the method under consideration cannot provide the robust QoS guarantee with finite power. It is clear from Fig. 6 that the proposed robust Tomlinson–Harashima designs are capable of (robustly) satisfying the SINR requirements for larger values of uncertainty sizes than the other approaches. It is also apparent that they expend less power in doing so.

VIII. CONCLUSION

We have presented a unified approach to the design of robust linear and nonlinear transceivers with user-specified QoS requirements subject to a deterministically bounded channel uncertainty model. The proposed approach formulated the QoS requirements in terms of MSE constraints and showed that these constraints imply corresponding constraints on the achieved SINR of each user. Our approach provided (convex) semidefinite program formulations of the design problem that can be efficiently solved. Furthermore, these design formulations were obtained for a rather general model of bounded channel uncertainty that includes many common uncertainty regions. We also showed how these designs can be used to provide quasi-convex formulations for the robust counterpart of the problem of fair transceiver design that maximizes the signal quality of the user with the weakest signal. Numerical results

demonstrated that under bounded uncertainty conditions, the proposed designs provided guaranteed satisfaction of a larger set of QoS requirements than the existing approaches that have considered bounded uncertainty models, and that they require less transmission power in order to satisfy these requirements.

APPENDIX A
PROOF OF LEMMA 1

Consider the quantity $(\hat{u}_k - u_k)$ in (3), where $u_k = s_k + i_k$. When the modulo operator at the receiver correctly removes i_k from $\hat{u}_k$, then $\hat{s}_k - s_k = \hat{u}_k - u_k$ and hence

$$\hat{s}_k - s_k = (g_k \mathbf{h}_k \mathbf{P} - \mathbf{m}_k - \mathbf{b}_k)\mathbf{v} + g_k n_k \quad (26)$$

or, equivalently

$$\hat{s}_k = s_k + \sum_{j=1}^{K} \alpha_j v_j + g_k n_k \quad (27)$$

where the α_js have been implicitly defined. By construction

$$v_k = s_k - \sum_{j<k} b_{k,j} v_j + i_k. \quad (28)$$

Therefore, under the assumption that i_k is removed correctly by the modulo operation, the input to the decision device of user k can be written as

$$\hat{s}_k = (1 + \alpha_k) s_k + \sum_{j<k}(\alpha_j - \alpha_k b_{k,j}) v_j + \sum_{j>k} \alpha_j v_j + g_k n_k$$
$$= a_k s_k + \sum_{j\neq k} a_j v_j + a_0 n_k \quad (29)$$

where we have implicitly defined the a_js. The decision point SINR is

$$\text{SINR}_k = \text{E}\left\{ \frac{|a_k|^2 |s_k|^2}{\left| \sum_{j<k} a_j v_j + \sum_{j>k} a_j v_j \right|^2 + |a_0|^2 \sigma_k^2} \right\}. \quad (30)$$

This expression can be simplified by observing from (28) that for $j < k$, v_j is independent of s_k. (This is a straightforward consequence of the sequential nature of the THP.) Furthermore, since we make the standard assume that v_j is independent identically distributed [28, Th. 3.1], v_j, $j > k$, is also independent of s_k. (This can be verified by a simple contradiction argument.) Combining this result with the assumption of negligible precoding loss $\text{E}\{\mathbf{v}\mathbf{v}^H\} = \mathbf{I}$, e.g., [27]–[29], we have that

$$\text{SINR}_k = \text{E}\left\{ \frac{|a_k|^2}{\sum_{j\neq k} |a_j|^2 + |a_0|^2 \sigma_k^2} \right\}. \quad (31)$$

From (29), we also have that

$$\text{E}\{|\hat{s}_k|^2\} = |a_k|^2 + \sum_{i\neq k} |a_i|^2 + |a_0|^2 \sigma_k^2. \quad (32)$$

Using the above, we can write MSE_k as

$$\text{E}\{|\hat{s}_k - s_k|^2\} = |a_k - 1|^2 + \sum_{i\neq k} |a_i|^2 + |a_0|^2 \sigma_k^2$$
$$= \text{E}\{|\hat{s}_k|^2\} + 1 - 2\text{Re}\{a_k\}$$

and we can write

$$1 + \frac{1}{\text{SINR}_k} = \frac{\text{E}\{|\hat{s}_k|^2\}}{|a_k|^2}.$$

Now, consider the MSE constraint $\text{E}\{|\hat{s}_k - s_k|^2\} = \text{E}\{|\hat{s}_k|^2\} + 1 - 2\text{Re}\{a_k\} \leq \zeta_k \leq 1$. This can be written as

$$\text{E}\{|\hat{s}_k|^2\}(1 - \zeta_k) \leq 2\text{Re}\{a_k\}(1 - \zeta_k) - (1 - \zeta_k)^2$$
$$= \text{Re}^2\{a_k\} - \left(\text{Re}\{a_k\} - (1 - \zeta_k)\right)^2$$
$$\leq |a_k|^2.$$

The latter inequality is equivalent to $1 + 1/\text{SINR}_k \leq 1/(1 - \zeta_k)$ or, equivalently, $\text{SINR}_k \geq (1/\zeta_k) - 1$.

APPENDIX B
DERIVATION OF DESIGN FORMULATIONS 1 AND 2

The derivations are based on the following lemma, which is a concatenation of two results in [37].

Lemma 3: Consider the SOC constraint $\|\mathbf{A}\mathbf{x} + \mathbf{b}\| \leq y$ for every $[\mathbf{A}, \ \mathbf{b}]$ in the uncertainty region given by

$$\mathcal{U} = \left\{ [\mathbf{A}, \ \mathbf{b}] \,\middle|\, [\mathbf{A}, \ \mathbf{b}] = [\mathbf{A}^0, \ \mathbf{b}^0] \right.$$
$$+ \sum_{j=1}^{J} \theta_j [\mathbf{A}^j, \ \mathbf{b}^j], \boldsymbol{\theta} \in \mathcal{V} \Big\}$$
$$\mathcal{V} = \left\{ \boldsymbol{\theta} \,\middle|\, \boldsymbol{\theta}^T \mathbf{Q}^\ell \boldsymbol{\theta} \leq 1, \ell = 1, \ldots, L \right\} \quad (33)$$

where $\mathbf{Q}^\ell \geq \mathbf{0}$. Then the set $\mathcal{S}_1$ of pairs $(\mathbf{x}, y)$ satisfying $\|\mathbf{A}\mathbf{x} + \mathbf{b}\| \leq y$ for every $[\mathbf{A}, \ \mathbf{b}] \in \mathcal{U}$ is a subset of the set $\mathcal{S}_2$ of pairs $(\mathbf{x}, y)$ such that there exist nonnegative scalars $\mu^1, \ldots, \mu^L$ satisfying

$$\begin{bmatrix} y - \sum_{\ell=1}^{L} \mu^\ell & \mathbf{0} & (\mathbf{A}^0 \mathbf{x} + \mathbf{b}^0)^T \\ \mathbf{0} & \sum_{\ell=1}^{L} \mu^\ell \mathbf{Q}^\ell & \mathbf{z}(\mathbf{x})^T \\ \mathbf{A}^0 \mathbf{x} + \mathbf{b}^0 & \mathbf{z}(\mathbf{x}) & y\mathbf{I} \end{bmatrix} \geq \mathbf{0} \quad (34)$$

where $\mathbf{z}(\mathbf{x}) = [\mathbf{A}^1 \mathbf{x} + \mathbf{b}^1, \ldots, \mathbf{A}^J \mathbf{x} + \mathbf{b}^J]$. When $L = 1$, $\mathcal{S}_1 = \mathcal{S}_2$. □

To derive Design Formulation 1, we use the channel uncertainty model in (13) to write the left-hand side of each MSE constraint in (14c) as follows:

$$[\mathbf{h}_k \mathbf{P} - f_k \mathbf{m}_k - \mathbf{b}_k, \ \sigma_{n_k}]$$
$$= [\hat{\mathbf{h}}_k \mathbf{P} - f_k \mathbf{m}_k - \mathbf{b}_k, \ \sigma_{n_k}] + [\sum_{j=1}^{J} w_k^{(j)} \boldsymbol{\phi}_k^{(j)} \mathbf{P}, \ 0]$$
$$= [\hat{\mathbf{h}}_k \mathbf{P} - f_k \mathbf{m}_k - \mathbf{b}_k, \ \sigma_{n_k}] + \sum_{j=1}^{J} \theta_k^{(j)} [\delta_k \boldsymbol{\phi}_k^{(j)} \mathbf{P}, \ 0]$$
$$(35)$$

where $\theta_k^{(j)} = w_k^{(j)}/\delta_k$; hence $\boldsymbol{\theta}_k^T \mathbf{Q} \boldsymbol{\theta}_k \leq 1$. By comparing (35) to (33), we can invoke Lemma 3 with $L = 1$ to show the equivalence between the SOC constraints in (14c) and the corresponding LMIs in (17c). The nonnegativity constraints on each μ_k are implied by positive semidefiniteness of the diagonal blocks of the matrices in (17c). The derivation of Design Formulation 2 is similar, but when $L \geq 2$, the application of Lemma 3 results in a conservative design formulation, and hence an upper bound on the required transmission power.

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